On the Fixed Charge Transportation Problem

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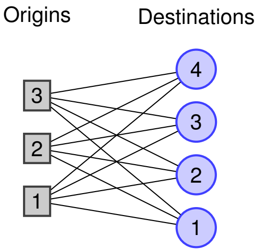
ROUTE 2014 - June 1st, 2014

The Fixed Charge Transportation Problem (FCTP)

- $S = \{1, 2, \ldots, m\}$ set of *m origins*
- $T = \{1, 2, \ldots, n\}$ set of *n destinations*
- $a_i \in \mathbb{N}$ supply at $i \in S$
- $b_j \in \mathbb{N}$ demand of $j \in T$
- c_{ij} unit cost for shipping from $i \in S$ to $j \in T$
- f_{ij} fixed cost for serving $j \in T$ from $i \in S$

- Assumption, w.l.o.g., $\sum_{i \in S} a_i = \sum_{j \in T} b_j$

Bipartite graph
$G = (S, T, E)$, where
$E = \{\{i, j\} : i \in S, j \in T\}$



Objective

Minimize the total fixed and variable cost for serving all destinations

Basic Formulation of the FCTP

- $x_{ij} \in \mathbb{R}_+$ flow along $\{i,j\} \in E$
- $y_{ij} \in \{0,1\}$ use of $\{i,j\} \in E$
- $m_{ij} = \min\{a_i, b_j\}$, $\{i,j\} \in E$

$$
\begin{array}{llll}
(\text{F0}) & \min & \sum_{\{i,j\}\in E} (c_{ij}x_{ij} + f_{ij}y_{ij}) & & (1) \\
& \text{s.t.} & \sum_{j\in T} x_{ij} = a_i & i \in S & (2) \\
& & \sum_{i\in S} x_{ij} = b_j & j \in T & (3) \\
& & 0 \le x_{ij} \le m_{ij}y_{ij} & \{i,j\} \in E & (4) \\
& & y_{ij} \in \{0,1\} & \{i,j\} \in E & (5)
\end{array}
$$

- LF0 linear relaxation of F0 (z(LF0) its optimal solution cost)

Observation

Any optimal FCTP solution is a *basic feasible solution* of (2)-(3)

Literature Review on Exact Algorithms

- Branch-and-bound
  - Kennigton and Unger, *Management Science* (1976)
  - Barr, Glover and Kligman, *Operations Research* (1981)
  - Cabot and Erenguc, *Naval Research Logistics Quarterly* (1984)
  - Cabot and Erenguc, *Management Science* (1986)
  - Palekar, Karwan and Zionts, *Management Science* (1990)
  - Lamar and Wallace, *Management Science* (1997)
- Branch-and-cut
  - Agarwal and Aneja, *Operations Research* (2012)
- Branch-and-cut-and-price
  - Roberti, Bartolini and Mingozzi, *Management Science* (2014) - forthcoming

Real-Life Applications of the FCTP

- Allocation of launch vehicles to space missions: Stroup, *Operations Research* (1967)
- Process selection: Hirsch and Dantzig, *Naval Research Logistics Quarterly* (1968)
- Solid-waste management: Walker, *Management Science* (1976)
- Teacher assignment: Hultberg and Cardoso, *European Journal of Operational Research* (1997)
- Distribution and transportation systems: Adlakha and Kowalski, *Omega* (2003)

Single-Pattern Formulation

Roberti, Bartolini and Mingozzi (2014)

An origin pattern of $i \in S$ is a vector $\boldsymbol{w} \in \mathbb{Z}_+^n$ s.t.

$$w_1 + w_2 + \ldots + w_n = a_i$$

and

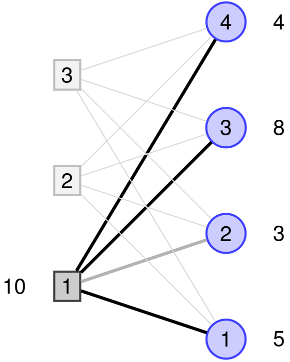
$$0 \le w_j \le m_{ij} \quad j \in T$$

W_i: index set of all origin patterns of $i \in S$

W: set of all origin patterns ($W = \bigcup_{i \in S} W_i$)

d_ℓ: cost of origin pattern $\ell \in W_i$

$$d_\ell = \sum_{j \in T : w_j^\ell > 0} (c_{ij} w_j^\ell + f_{ij})$$



$\boldsymbol{w} = [5\ 0\ 1\ 4]$

$d_\ell = f_{11} + 5c_{11} + f_{13} + c_{13} + f_{14} + 4c_{14}$

$\xi_\ell \in \{0, 1\}$ for origin pattern $\ell \in W$

$$
\begin{aligned}
\text{(F1)} \qquad \min \quad & \sum_{\ell \in W} d_\ell \xi_\ell && && (6)\\
\text{s.t.} \quad & \sum_{\ell \in W} w_j^\ell \xi_\ell = b_j && j \in T && (7)\\
& \sum_{\ell \in W_i} \xi_\ell = 1 && i \in S && (8)\\
& \xi_\ell \in \{0, 1\} && \ell \in W && (9)
\end{aligned}
$$

LF1 linear relaxation of F1 ($z(\text{LF1})$ its optimal solution cost)

- v_j dual variable of constraint (7) for $j \in T$
- u_i dual variable of constraint (8) for $i \in S$

Proposition

$z(\text{LF1}) \geq z(\text{LF0})$

Single-Pattern Formulation

Pricing Problem

For each $i \in S$, the pricing problem PR_i is a Multiple Choice Knapsack

- For each $j \in T$ and each flow $q = 1, \ldots, m_{ij}$, let $\varphi_{jq} \in \{0, 1\}$

$$
\begin{aligned}
(\text{PR}_\text{i}) \qquad \min \quad & \sum_{j \in T} \sum_{q=1}^{m_{ij}} (f_{ij} + q(c_{ij} - v_j))\varphi_{j\sigma} - u_i \\
\text{s.t.} \quad & \sum_{j \in T} \sum_{q=1}^{m_{ij}} q\varphi_{jq} = a_i \\
& \sum_{q=1}^{m_{ij}} \varphi_{jq} \le 1 && j \in T \\
& \varphi_{jq} \in \{0, 1\} && j \in T \quad q = 1, \ldots, m_{ij}
\end{aligned}
$$

- PR_i can be solved by dynamic programming recursions

Single-Pattern Formulation

Reversing Origins and Destinations

- The FCTP is symmetric in the origins and destinations
- Lower bound $z(\text{LF1})$ can also be computed by reversing origins and destinations
- Hereafter, we assume that origins and destinations are reversed if this provides a better lower bound $z(\text{LF1})$

Single-Pattern Formulation

Valid Inequalities

Lower bound $z(\text{LF1})$ can be tightened with the following cuts

- Set Covering: Agarwal and Aneja (2012)
- Chvátal-Gomory: Gomory (1958), Chvátal (1973)
- Lifted Chvátal-Gomory NEW
- Extended Generalized Upper Bound Cover: Gu et al. (1998)
- Feasibility NEW
- Couple NEW

Only robust cuts are considered

- $\overline{\text{LF1}}$ problem LF1 plus all valid inequalities mentioned ($z(\overline{\text{LF1}})$ its optimal solution cost)

Single-Pattern Formulation

Outline of the Exact Algorithm of Roberti et al. (2014)

- Computes lower bound $z(\overline{\text{LF1}})$ at the root node
- Cuts separated at the root node only
- Branching on edges $\{i, j\} \in E$
- Nodes explored with best-bound strategy

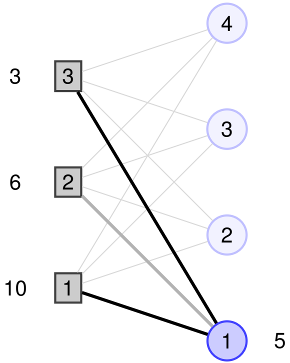
Double-Pattern Formulation

A destination pattern of $j \in T$ is a vector $\overline{\boldsymbol{w}} \in \mathbb{Z}_+^m$ s.t.

$$\overline{w}_1 + \overline{w}_2 + \ldots + \overline{w}_m = b_j$$

and

$$0 \leq \overline{w}_i \leq m_{ij} \quad i \in S$$



$\overline{\boldsymbol{w}} = [2\ 0\ 3]$

$\overline{d}_\ell = f_{11} + 2c_{11} + f_{31} + 3c_{31}$

$\overline{W}_j$: index set of all destination patterns of $j \in T$

$\overline{W}$: set of all destination patterns ($\overline{W} = \cup_{j \in T} \overline{W}_j$)

$\overline{d}_\ell$: cost of destination pattern $\ell \in \overline{W}_j$

$$\overline{d}_\ell = \sum_{i \in S : \overline{w}_i^\ell > 0} (c_{ij} \overline{w}_i^\ell + f_{ij})$$

$\xi_\ell \in \{0,1\}$ for $\ell \in W$, and $\bar{\xi}_\ell \in \{0,1\}$ for $\ell \in \overline{W}$

$$
\begin{aligned}
\text{(F2)} \qquad \frac{1}{2}\min \quad & \sum_{\ell \in W} d_\ell \xi_\ell + \sum_{\ell \in \overline{W}} \bar{d}_\ell \bar{\xi}_\ell \\
\text{s.t.} \quad & \sum_{\ell \in W_i} \xi_\ell = 1 && i \in S \\
& \sum_{\ell \in \overline{W}_j} \bar{\xi}_\ell = 1 && j \in T \\
& \sum_{\ell \in W_i} w_j^\ell \xi_\ell = \sum_{\ell \in \overline{W}_j} \bar{w}_i^\ell \bar{\xi}_\ell && \{i,j\} \in E \\
& \sum_{\ell \in W} d_\ell \xi_\ell = \sum_{\ell \in \overline{W}} \bar{d}_\ell \bar{\xi}_\ell \\
& \xi_\ell \in \{0,1\} \;\; \ell \in W \qquad \bar{\xi}_\ell \in \{0,1\} \;\; \ell \in \overline{W}
\end{aligned}
$$

LF2 linear relaxation of F2 (z(LF2) its optimal solution cost)

Proposition

$z(\text{LF2}) \geq z(\text{LF1}) \geq z(\text{LF0})$

Double-Pattern Formulation

Valid Inequalities and Comparison with the Single-Pattern Formulation

$z(\text{LF2})$ can be tightened with the following Edge-Quantity (EQ) equalities

$$\sum_{\ell \in W_i : w_j^\ell = q} \xi_\ell = \sum_{\ell \in \overline{W}_j : \overline{w}_i^\ell = q} \overline{\xi}_\ell \qquad \{i,j\} \in E \quad q = 1, \ldots, m_{ij}$$

- $\overline{\text{LF2}}$ problem LF2 plus EQs ($z(\overline{\text{LF2}})$ its optimal solution cost)
- Problem $\overline{\text{LF2}}$ implies the following cuts derived for F1
  - Chvátal-Gomory
  - Lifted Chvátal-Gomory
  - Extended Generalized Upper Bound Cover
  - Feasibility
  - Couple

Proposition

$z(\overline{\text{LF2}}) \geq \left(z(\overline{\text{LF1}}) \text{ without Set Covering inequalities} \right)$

Double-Pattern Formulation

Outline of the New Exact Branch-and-Cut-and-Price Algorithm

- Lower bound at the root node given by $z(\overline{\text{LF2}})$
  - EQs separated by inspection
  - Pricing problem solved with dynamic programming recursions
- Cuts are separated at each node of the search tree
- Branching on edges $\{i, j\} \in E$
- Nodes explored with best-bound strategy

- 390 instances
  - 30 proposed by Agarwal and Aneja (2012): 3 classes of 10 instances each
  - 180 proposed by Roberti et al. (2014): 18 classes of 10 instances each
  - 180 new instances: 18 classes of 10 instances each
- All 390 instances feature
  - $m = n$
  - f_{ij} random in $[200, 800]$
  - Variable costs that amount for θ% of the total cost, with $\theta = 0, 20, 50$
  - a_i and b_j random in the range $[1, B]$, with $B = 20, 50, 100$

Computational Results

Instances proposed by Agarwal and Aneja (2012)

n = 15 and B = 20

	Cplex 12.5				AA12			Single-Pattern					Double-Pattern				
θ	LF0	$\overline{\text{LF0}}$	Opt	T	LBr	Opt	T	LF1	$\overline{\text{LF1}}$	Nds	Opt	T	LF2	$\overline{\text{LF2}}$	Nds	Opt	T
0	74.9	91.5	10	20	89.6	10	139	84.0	99.5	9	10	0.8	88.8	99.9	1	10	0.2
20	81.9	95.3	10	36	92.5	8	51	88.2	99.6	11	10	0.5	92.9	99.9	1	10	0.2
50	85.7	96.4	10	18	93.6	9	61	92.3	99.7	12	10	0.6	94.0	99.8	2	10	0.2
	80.8	**94.4**	**30**	**25**	**91.9**	**27**	**87**	**88.2**	**99.6**		**30**	**0.6**	**91.9**	**99.9**		**30**	**0.2**

AA12: Agarwal and Aneja (2012) - 600 secs of time limit - Cplex 11.2 - machine $\approx 2\times$ slower than ours

Computational Results

Instances proposed by Roberti et al. (2014)

B = 20

		Cplex 12.5				Single-Pattern					Double-Pattern				
n	θ	LF0	$\overline{\text{LF0}}$	Opt	T	LF1	$\overline{\text{LF1}}$	Nds	Opt	T	LF2	$\overline{\text{LF2}}$	Nds	Opt	T
30	0	81.8	94.2	4	3221	88.5	99.1	234	10	13	93.2	99.4	20	10	2
30	20	84.7	95.0	6	4784	90.4	99.4	176	10	11	93.7	99.5	19	10	2
30	50	87.1	96.2	8	1850	91.9	99.5	90	10	9	94.4	99.6	11	10	2
50	0	82.8	94.7	0	-	89.7	99.3	2110	10	116	94.3	99.5	123	10	24
50	20	85.5	95.0	0	-	90.6	99.4	1109	10	95	94.4	99.6	71	10	18
50	50	88.7	96.3	0	-	93.3	99.5	7489	10	324	95.5	99.7	161	10	39
70	0	85.0	95.3	0	-	90.7	99.5	3251	10	421	95.3	99.6	180	10	76
70	20	86.8	95.6	0	-	91.4	99.5	12830	9	1055	95.5	99.6	429	10	169
70	50	89.6	96.4	0	-	93.3	99.6	14306	10	1395	96.2	99.7	353	10	166
		85.8	**95.4**	**18**		**91.1**	**99.4**		**89**	**375**	**94.7**	**99.6**		**90**	**55**

10800 seconds of time limit

Computational Results

Instances proposed by Roberti et al. (2014)

B = 50

		Cplex 12.5				Single-Pattern					Double-Pattern				
n	θ	LF0	$\overline{\text{LF0}}$	Opt	T	LF1	$\overline{\text{LF1}}$	Nds	Opt	T	LF2	$\overline{\text{LF2}}$	Nds	Opt	T
20	0	73.6	89.8	4	4569	83.1	98.1	1306	10	37	87.7	99.1	16	10	4
20	20	79.2	92.8	7	2269	86.2	98.5	444	10	16	90.2	99.4	8	10	2
20	50	84.1	94.9	10	1234	90.0	99.0	282	10	13	92.7	99.7	4	10	2
30	0	76.6	90.4	0	-	83.8	98.3	5497	10	353	89.1	99.1	44	10	16
30	20	78.0	90.6	0	-	85.0	98.3	11264	10	899	89.2	99.2	49	10	22
30	50	83.9	93.6	0	-	89.0	98.7	5676	10	387	91.9	99.5	34	10	13
40	0	77.6	90.4	0	-	84.3	98.0	51214	5	5060	89.9	99.0	269	10	153
40	20	80.4	92.0	0	-	87.2	98.4	18443	6	1793	91.1	99.1	434	10	265
40	50	84.8	93.6	0	-	89.8	98.7	13870	3	1741	92.5	99.3	216	10	125
		79.8	**92.0**	**21**		**86.5**	**98.4**		**74**	**788**	**90.5**	**99.3**		**90**	**67**

10800 seconds of time limit

(Preliminary) Computational Results

New Instances

B = 20

		Double-Pattern			
n	θ	$\overline{\text{LF2}}$	Nds	Opt	T
100	0	99.6	1289	10	925
100	20	99.7	1765	10	1721
100	50	99.7	1614	10	1840
120	0	99.7	4115	10	5156
120	20	99.7	4159	9	4581
120	50	99.7	5058	10	9679

B = 50

		Double-Pattern			
n	θ	$\overline{\text{LF2}}$	Nds	Opt	T
50	0	99.1	1579	10	1191
50	20	99.3	656	10	503
50	50	99.4	351	10	340
60	0	99.2	2583	10	3960
60	20	99.3	2081	10	4180
60	50	99.4	1181	10	1771

B = 100

		Double-Pattern			
n	θ	$\overline{\text{LF2}}$	Nds	Opt	T
40	0	98.7	962	10	1981
40	20	99.0	424	10	726
40	50	99.5	123	10	197
50	0	98.8	3319	10	12617
50	20	99.1	2378	10	6956
50	50	99.2	891	10	2765

Bibliography I

[Adlakha and Kowalski 2003] V. Adlakha and K. Kowalski
A simple heuristic for solving small Fixed-Charge Transportation Problems
Omega 31(3) 205-211. 2003

[Agarwal and Aneja 2012] Y. Agarwal and Y. Aneja
Fixed-charge transportation problem: Facets of the projection polyhedron
Operations Research 60(3) 638-654. 2012

[Barr et al. 1981] R. S. Barr, F. Glover, and D. Klingman
A new optimization method for large scale fixed charge transportation problems
Operations Research 29(3) 448-463. 1981

[Cabot and Erenguc 1984] A. V. Cabot and S. S. Erenguc
Some branch and bound procedures for fixed cost transportation problems
Naval Research Logistics Quarterly 31(1) 145-154. 1984

[Cabot and Erenguc 1986] A. V. Cabot and S. S. Erenguc
Improved penalties for fixed cost linear programs using lagrangean relaxation
Management Science 32(1) 856-869. 1986

[Chvátal 1973] V. Chvátal
Edmonds polytopes and a hierarchy of combinatorial problems
Discrete Mathematics 4(4) 305-337. 1973

Bibliography II

[Gomory 1958] R. E. Gomory
Outline of an algorithm for integer solutions to linear programs
Bulletin of the AMS 4 275-278. 1958

[Gu et al. 1998] Z. Gu, G. L. Nemhauser, and M. W. P. Savelsbergh
Lifted cover inequalities for 0-1 integer programs: Computation
INFORMS Journal on Computing 10(4) 427-437. 1998

[Hirsch and Dantzig 1968] W. M. Hirsch and G. B. Dantzig
The Fixed Charge Problem
Naval Research Logistics Quarterly 15 413-424. 1968

[Hultberg and Cardoso 1997] T. H. Hultberg and D. M. Cardoso
The Teacher Assignment Problem: A special case of the Fixed Charge Transportation Problem
European Journal of Operational Research 101(3) 463-473. 1997

[Kennigton and Unger 1976] J. Kennigton and E. Unger
A new branch-and-bound algorithm for the fixed-charge transportation problem
Management Science 22(10) 1116-1126. 1976

[Lamar and Wallace 1997] B. W. Lamar and C. A. Wallace
Revised-modified penalties for fixed charge transportation problems
Management Science 43(10) 1431-1436. 1997

Bibliography III

[Palekar et al. 1990] U. S. Palekar, M. H. Karwan, and S. Zionts
A branch and bound method for the fixed charge transportation problem
Management Science 36(9) 1092-1105. 1990

[Roberti et al. 2014] R. Roberti, E. Bartolini, A. Mingozzi.
The Fixed Charge Transportation Problem: An exact algorithm based on a new integer programming formulation
Management Science (forthcoming). 2014.

[Stroup 1967] J. W. Stroup
Allocation of launch vehicles to space missions: a Fixed-Cost Transportation Problem
Operations Research 15(6) 1157-1163. 1967

[Walker 1976] W. E. Walker
A heuristic adjacent extreme point algorithm for the fixed charge problem
Management Science 22(5) 587-596. 1976