Dual-guided pivot rules for LP
+
Vector Space Decomposition for LP

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Route 2014, Snekkersten, Denmark
June 1-4, 2014

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Motivation

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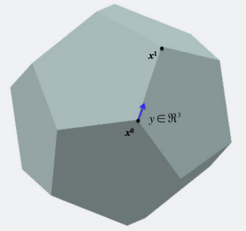
... and Dantzig-Wolfe decompostion ?

Observation #1

LP in standard form

$$\begin{aligned} \min \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} & \\ & \mathbf{A}\mathbf{x} = \mathbf{b} & [\pi] \\ & \mathbf{x} \geq \mathbf{0} & \end{aligned}$$



From $\mathbf{x}^0$ to $\mathbf{x}^1$

1. Find a *potential* improving direction $\mathbf{y}^0 \in \mathbb{R}^n$.
2. Determine step-size $\rho^0 \in \mathbb{R}$.
3. Compute $\mathbf{x}^1 := \mathbf{x}^0 + \rho^0 \mathbf{y}^0$.

$\bar{c}_j = 0, \forall j \in B$: PRICING FOR $j \in N$ (NON-BASIC VARIABLES)

Selection of an entering variable into basis $\mathbf{A}_B$ relies on
the minimum reduced cost of non-basic variables

$$\boldsymbol{\pi}^\intercal = \mathbf{c}_B^\intercal \mathbf{A}_B^{-1} \qquad \bar{c}_j = c_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j, \quad \forall j \in N.$$

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Find the minimum reduced cost value μ (Optimal if $\mu \geq 0$)

$$\begin{aligned} \max \quad & \mu \\ & \mu \leq c_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j, \quad \forall j \in N \qquad [y_j] \end{aligned}$$

*** μ is the smallest reduced cost (given $\boldsymbol{\pi}$). ***

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FIND THE MINIMUM REDUCED COST VALUE μ (OPTIMAL IF $\mu \geq 0$)

$$\begin{aligned} \max \quad & \mu \\ & \mu \leq c_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j, \quad \forall j \in N \qquad [y_j] \end{aligned}$$

*** μ is the smallest reduced cost (given $\boldsymbol{\pi}$). ***

EQUIVALENT TO FINDING A CONVEX COMBINATION OF NON-BASIC VARIABLES

$$\begin{aligned} \mu = \quad \min \quad & \sum_{j \in N} \bar{c}_j y_j \\ & \sum_{j \in N} y_j = 1 \qquad [\mu] \\ & y_j \geq 0, \quad \forall j \in N \end{aligned}$$

Direction $\mathbf{y}^0 \in \mathbb{R}^n$

$\mathbf{y}^0$ in the Primal Simplex

Direction $\mathbf{y}^0 \in \mathbb{R}^n$:

the selected entering variable,

the non-selected non-basic variables (they remain at 0),

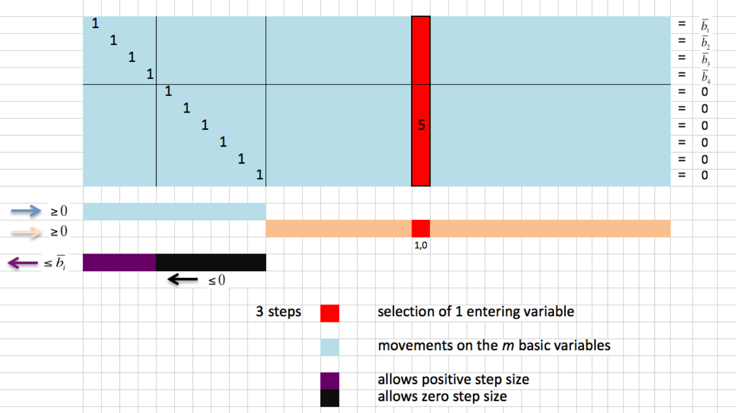
and the basic ones.

Step size computed such that $\mathbf{x}^1 := \mathbf{x}^0 + \rho^0\, \mathbf{y}^0 \geq \mathbf{0}$.

Left multiplication by basis inverse $\mathbf{A}_B^{-1}$

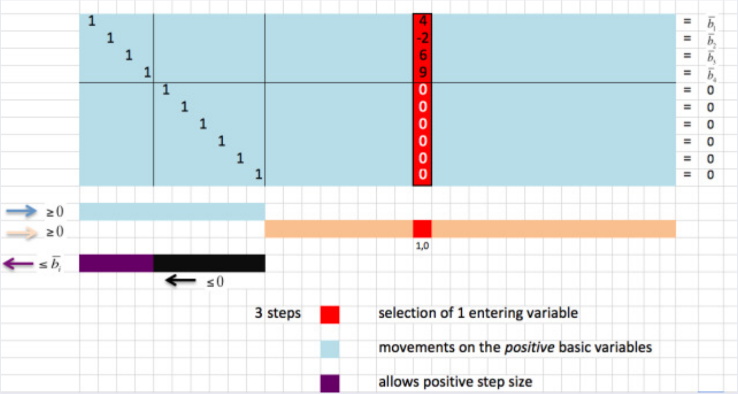
	x1	x2	x3	λ4	λ5	λ6	x4	x5	x6	x7	x8			ψ	θ_ω
c	2	3	1				10	17	-20	14	-4				-5
	1						2	2	1	-5	7	=	30	2	1
		1					4	3	-5	10	-10	=	25	3	-2
			1				-3	1	2	3	11	=	50	1	5
				1					6	5	-13	=	0	?	0
					1				3	4	-8	=	0	?	0
						1			3	-4	0	=	0	?	0
x^0	30	25	50	0	0	0	0	0	0	0	0	185	$= c^T x^0$		
									0.4	0.3	0.3				

Primal Simplex tableau



*** Changes on at most $m + 1$ components. ***

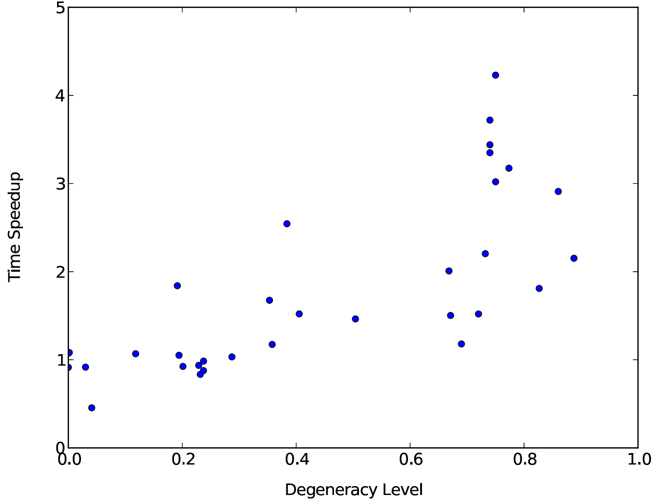
Positive Edge strategy



*** Changes on at most $p + 1$ components. ***
Non-degenerate pivot.

Positive Edge : identification in $O(m)$

Speedup > 2 for LPs with a degeneracy level above 25%



Vectors and matrices are written in **bold face**.

$\mathbf{I}_\ell$: the $\ell \times \ell$ identity matrix.
$\mathbf{0}$ ($\mathbf{1}$) : a vector/matrix with all zeros (ones) entries of appropriate dimensions.
$\mathbf{A}_{RC}$: sub-matrix of $\mathbf{A}$ containing the rows and columns indexed by R and C.

Basis $\mathbf{A}_B$, inverse $\mathbf{A}_B^{-1}$, $\mathbf{c}_B^\intercal \mathbf{x}_B$, $\mathbf{A}_B \mathbf{x}_B$, $\boldsymbol{\pi} = \mathbf{c}_B \mathbf{A}_B^{-1}$ ...
$\mathbf{l}_F < \mathbf{x}_F < \mathbf{u}_F$, $\mathbf{x}_L = \mathbf{l}_L$, $\mathbf{x}_U = \mathbf{u}_U$

USEFUL DECOMPOSITION OF $\mathbf{x} \in \mathbb{R}^n$ IN $\mathbf{Ax} = \mathbf{b}$, $\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_F \\ \mathbf{x}_L \\ \mathbf{x}_U \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{x}_F \\ \mathbf{x}_{B_L} \\ \mathbf{x}_{B_U} \\ \mathbf{x}_{N_L} \\ \mathbf{x}_{N_U} \end{bmatrix}$$

$$\text{For } \emptyset \subseteq S \subseteq B : \mathbf{x} = \begin{bmatrix} \mathbf{x}_S \\ \mathbf{x}_{\bar{S}} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{S_F} \\ \mathbf{x}_{S_L} \\ \mathbf{x}_{S_F} \\ \mathbf{x}_{\bar{S}_U} \\ \mathbf{x}_{\bar{S}_L} \\ \mathbf{x}_{\bar{S}_U} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \mathbf{x}_\emptyset \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_F \\ \mathbf{x}_L \\ \mathbf{x}_U \end{bmatrix}$$

Generic Algorithm with single parameter set S, $\emptyset \subseteq S \subseteq B$

1. Let $k = 0$ and assume a feasible basic solution $\mathbf{x}^k$ to LP (standard form).
2. Select $S \subseteq B$.

$$\begin{aligned} z^\star := \quad & \min \quad \mathbf{c}_S^\mathsf{T}\mathbf{x}_S + \mathbf{c}_{\bar{S}}^\mathsf{T}\mathbf{x}_{\bar{S}} \\ \text{st.} \quad & \mathbf{A}_S^\mathsf{T}\mathbf{x}_S + \mathbf{A}_{\bar{S}}^\mathsf{T}\mathbf{x}_{\bar{S}} = \mathbf{b} \\ & \mathbf{x}_S, \mathbf{x}_{\bar{S}} \geq \mathbf{0} \end{aligned}$$

3. From $\mathbf{A}_S$, retrieve an $s \times s$ basis $\mathbf{A}_{RS}$ for row-set R.

$$\begin{aligned} z^\star := \quad & \min \quad \mathbf{c}_S^\mathsf{T}\mathbf{x}_S + \mathbf{c}_{\bar{S}}^\mathsf{T}\mathbf{x}_{\bar{S}} \\ \text{st.} \quad & \mathbf{A}_{RS}^\mathsf{T}\mathbf{x}_S + \mathbf{A}_{R\bar{S}}^\mathsf{T}\mathbf{x}_{\bar{S}} = \mathbf{b}_R \quad [\boldsymbol{\pi}_R] \\ & \mathbf{A}_{ZS}^\mathsf{T}\mathbf{x}_S + \mathbf{A}_{Z\bar{S}}^\mathsf{T}\mathbf{x}_{\bar{S}} = \mathbf{b}_Z \quad [\boldsymbol{\pi}_Z] \\ & \mathbf{x}_S, \mathbf{x}_{\bar{S}} \geq \mathbf{0} \end{aligned}$$

4. Fix $\boldsymbol{\pi}_R$ in row-set R ; $\boldsymbol{\pi}_Z$ is free.
5. Determine the smallest reduced cost μ_S^k.

$$\mu_S^k := \max \ \mu \quad \text{st.} \ \mu \leq c_j - \boldsymbol{\pi}_R^\mathsf{T}\mathbf{a}_{Rj} - \boldsymbol{\pi}_Z^\mathsf{T}\mathbf{a}_{Zj}, \quad \forall j$$

   If $\mu_S^k \geq 0$, STOP. Current solution $\mathbf{x}^k$ is optimal for LP.
6. Retrieve direction $\mathbf{y}_S^k \in \mathbb{R}^n$ and compute its maximum step-size ρ_S^k.
7. Update $\mathbf{x}^{k+1} := \mathbf{x}^k + \rho_S^k \mathbf{y}_S^k$; $z^{k+1} := z^k + \rho_S^k \mu_S^k$; $k := k + 1$.
8. Goto Step 2.

Linear program *LP*

$$
\begin{aligned}
z^\star := \quad & \min \quad \mathbf{c}^\mathsf{T}\mathbf{x} \\
\text{st.} \quad & \qquad \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}
\end{aligned}
$$

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\mathbf{l} \leq \ & \mathbf{x} \leq \mathbf{u}
\end{aligned}
$$

Generic Algorithm with single parameter set S, $\emptyset \subseteq S \subseteq B$

1. Let $k = 0$ and assume a feasible basic solution $\mathbf{x}^k$ to LP.
2. Select $\emptyset \subseteq S \subseteq B$.
3. From $\mathbf{A}_S$, retrieve $\mathbf{A}_{RS}$ and construct *residual problem* $\mathrm{LP}_S(\mathbf{x}^k)$.
4. Fix dual variables in row-set R : $\boldsymbol{\pi}_R$.
5. Determine the value of the smallest reduced cost μ_S^k.
   If $\mu_S^k \geq 0$, STOP. Current solution $\mathbf{x}^k$ is optimal for LP.
6. Retrieve direction $\mathbf{y}_S^k \in \mathbb{R}^n$ and compute its maximum step-size ρ_S^k.
7. Update $\mathbf{x}^{k+1} := \mathbf{x}^k + \rho_S^k \, \mathbf{y}_S^k$;
   $z^{k+1} := z^k + \rho_S^k \, \mu_S^k$;
   $k := k + 1$.
8. Goto Step 2.

Assume a feasible solution $\mathbf{x}^k = \begin{bmatrix} \mathbf{x}_F^k \\ \mathbf{x}_L^k \\ \mathbf{x}_U^k \end{bmatrix}$

$$\mathbf{x} := \mathbf{x}^k + \mathbf{y}, \quad \mathbf{y} \in \mathbb{R}^n$$

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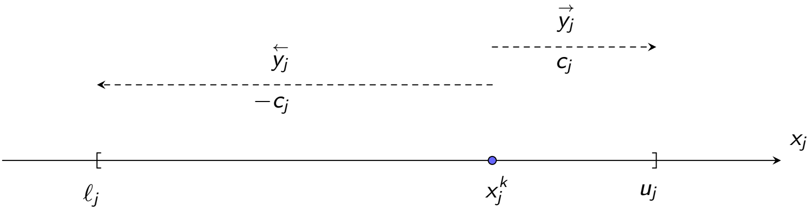
$$= \mathbf{x}^k + (\overrightarrow{\mathbf{y}} - \overleftarrow{\mathbf{y}}), \qquad \overrightarrow{\mathbf{y}}, \overleftarrow{\mathbf{y}} \geq \mathbf{0}, \quad \overrightarrow{\mathbf{y}}^\intercal \overleftarrow{\mathbf{y}} = 0, \quad \overrightarrow{\mathbf{y}} \leq \overrightarrow{\mathbf{r}}^k, \quad \overleftarrow{\mathbf{y}} \leq \overleftarrow{\mathbf{r}}^k$$

ASSUME A FEASIBLE SOLUTION $\mathbf{x}^k = \begin{bmatrix} \mathbf{x}_F^k \\ \mathbf{x}_L^k \\ \mathbf{x}_U^k \end{bmatrix}$

$$\mathbf{x} := \mathbf{x}^k + \mathbf{y}, \quad \mathbf{y} \in \mathbb{R}^n$$

$$= \mathbf{x}^k + (\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}), \qquad \vec{\mathbf{y}}, \overleftarrow{\mathbf{y}} \geq \mathbf{0}, \quad \vec{\mathbf{y}}^\mathsf{T} \overleftarrow{\mathbf{y}} = 0, \quad \vec{\mathbf{y}} \leq \vec{\mathbf{r}}^k, \quad \overleftarrow{\mathbf{y}} \leq \overleftarrow{\mathbf{r}}^k$$

$$= \mathbf{x}^k + \left(\begin{bmatrix} \vec{\mathbf{y}}_F \\ \vec{\mathbf{y}}_L \\ \vec{\mathbf{y}}_U \end{bmatrix} - \begin{bmatrix} \overleftarrow{\mathbf{y}}_F \\ \overleftarrow{\mathbf{y}}_L \\ \overleftarrow{\mathbf{y}}_U \end{bmatrix} \right); \quad \vec{\mathbf{y}}, \overleftarrow{\mathbf{y}} \geq \mathbf{0}, \quad \vec{\mathbf{y}}^\mathsf{T} \overleftarrow{\mathbf{y}} = 0, \quad \vec{\mathbf{y}} \leq \vec{\mathbf{r}}^k, \quad \overleftarrow{\mathbf{y}} \leq \overleftarrow{\mathbf{r}}^k$$



Residual problem $LP(\mathbf{x}^k)$; $\mathbf{x} := \mathbf{x}^k + (\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}})$ (change of variables)

$$\begin{aligned} z^\star := \min \quad & \mathbf{c}^\mathsf{T}\mathbf{x}^k + \mathbf{c}^\mathsf{T}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) \\ \text{st.} \quad & \mathbf{A}\mathbf{x}^k + \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \mathbf{b} \\ & \mathbf{0} \le \vec{\mathbf{y}} \le \vec{\mathbf{r}}^k \\ & \mathbf{0} \le \overleftarrow{\mathbf{y}} \le \overleftarrow{\mathbf{r}}^k \end{aligned}$$

Step 3 Construct residual problem $LP(\mathbf{x}^k)$

Residual problem $LP(\mathbf{x}^k)$; $\mathbf{x} := \mathbf{x}^k + (\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}})$ (change of variables)

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\text{st.} \quad & \mathbf{A}\mathbf{x}^k + \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \mathbf{b} \\
& \mathbf{0} \leq \vec{\mathbf{y}} \leq \vec{\mathbf{r}}^k \\
& \mathbf{0} \leq \overleftarrow{\mathbf{y}} \leq \overleftarrow{\mathbf{r}}^k
\end{aligned}
$$

Residual problem $LP(\mathbf{x}^k)$

$$
\begin{aligned}
z^\star := \mathbf{c}^\mathsf{T}\mathbf{x}^k + \min \quad & \mathbf{c}^\mathsf{T}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) \\
\text{st.} \quad & \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \mathbf{0} \\
& \mathbf{0} \leq \vec{\mathbf{y}} \leq \vec{\mathbf{r}}^k \\
& \mathbf{0} \leq \overleftarrow{\mathbf{y}} \leq \overleftarrow{\mathbf{r}}^k
\end{aligned}
$$

Residual problem with $S = F, \quad \bar{S} = L \cup U$

$z^\star := \mathbf{c}^\intercal \mathbf{x}^k +$

$$\begin{array}{lrcrcrcl}
\min & \mathbf{c}_F^\intercal(\vec{\mathbf{y}}_F - \overleftarrow{\mathbf{y}}_F) & + & \mathbf{c}_L^\intercal(\vec{\mathbf{y}}_L) & - & \mathbf{c}_U^\intercal(\overleftarrow{\mathbf{y}}_U) & & \\
\text{st.} & \mathbf{A}_F(\vec{\mathbf{y}}_F - \overleftarrow{\mathbf{y}}_F) & + & \mathbf{A}_L(\vec{\mathbf{y}}_L) & - & \mathbf{A}_U(\overleftarrow{\mathbf{y}}_U) & = & \mathbf{0} \\
& \vec{\mathbf{y}}_F \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_F \geq \mathbf{0}, & & \vec{\mathbf{y}}_L \geq \mathbf{0}, & & \overleftarrow{\mathbf{y}}_U \geq \mathbf{0} & & \\
& \vec{\mathbf{y}}_F \leq \vec{\mathbf{r}}_F,\ \overleftarrow{\mathbf{y}}_F \leq \overleftarrow{\mathbf{r}}_F, & & \vec{\mathbf{y}}_L \leq \vec{\mathbf{r}}_L, & & \overleftarrow{\mathbf{y}}_U \leq \overleftarrow{\mathbf{r}}_U & &
\end{array}$$

Residual problem with $S = B$ (basic), $\bar{S} = N$ (non-basic)

$$
\begin{array}{llllll}
\mathbf{c}^\mathsf{T}\mathbf{x}^k + & \min \ \mathbf{c}_B^\mathsf{T}(\vec{\mathbf{y}}_B - \overleftarrow{\mathbf{y}}_B) & + & \mathbf{c}_{N_L}^\mathsf{T}\vec{\mathbf{y}}_{N_L} - \mathbf{c}_{N_U}^\mathsf{T}\overleftarrow{\mathbf{y}}_{N_U} & & \\
& \text{st.} \ \ \mathbf{A}_B(\vec{\mathbf{y}}_B - \overleftarrow{\mathbf{y}}_B) & + & \mathbf{A}_{N_L}\vec{\mathbf{y}}_{N_L} - \mathbf{A}_{N_U}\overleftarrow{\mathbf{y}}_{N_U} & = & \mathbf{0} \\
& \vec{\mathbf{y}}_B \geq \mathbf{0}, \ \overleftarrow{\mathbf{y}}_B \geq \mathbf{0} & & \vec{\mathbf{y}}_{N_L} \geq \mathbf{0}, \ \overleftarrow{\mathbf{y}}_{N_U} \geq \mathbf{0} & & \\
& \vec{\mathbf{y}}_F \leq \vec{\mathbf{r}}_F, \ \overleftarrow{\mathbf{y}}_F \leq \overleftarrow{\mathbf{r}}_F & & & & \\
& \overleftarrow{\mathbf{y}}_{B_L} \leq \mathbf{0}, \ \vec{\mathbf{y}}_{B_U} \leq \mathbf{0} & & \vec{\mathbf{y}}_{N_L} \leq \vec{\mathbf{r}}_{N_L}, \overleftarrow{\mathbf{y}}_{N_U} \leq \overleftarrow{\mathbf{r}}_{N_U} & &
\end{array}
$$

$$B = F \cup B_L \cup B_U; \qquad N = N_L \cup N_U$$

For $\emptyset \subseteq S \subseteq B$, find $\mathbf{y}^k = \begin{bmatrix} \mathbf{y}^k_S \\ \mathbf{y}^k_{\bar{S}} \end{bmatrix} = \begin{bmatrix} (\vec{\mathbf{y}}^k_S - \overleftarrow{\mathbf{y}}^k_S) \\ (\vec{\mathbf{y}}^k_{\bar{S}} - \overleftarrow{\mathbf{y}}^k_{\bar{S}}) \end{bmatrix}$ of min reduced cost μ^k.

For basic columns $\mathbf{A}_S$,
retrieve a partial basis $\mathbf{A}_{RS}$, a set of s independent rows.

$$\mathbf{T} = \begin{bmatrix} \mathbf{A}_{RS} & \mathbf{0} \\ \mathbf{A}_{ZS} & \mathbf{I}_{m-s} \end{bmatrix} \qquad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}^{-1}_{RS} & \mathbf{0} \\ -\mathbf{A}_{ZS}\mathbf{A}^{-1}_{RS} & \mathbf{I}_{m-s} \end{bmatrix}$$

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Indeed, $\mathbf{T}$ is a basis of $\mathbb{R}^m$.

$$\emptyset \subseteq S \subseteq B$$

Perform $\mathbf{T}^{-1}\mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \mathbf{0}$.

A block angular structure appears *after* the transformation by $\mathbf{T}^{-1}$.

General case $\quad \pi^{\mathsf{T}} = \psi^{\mathsf{T}}\mathbf{T}^{-1}$

$$\begin{array}{llll}
\mathbf{c}^{\mathsf{T}}\mathbf{x}^k + \min & \mathbf{c}_S^{\mathsf{T}}(\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) \; + & \mathbf{c}_{\bar{S}}^{\mathsf{T}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & \\
\text{st.} & (\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) \; + & \bar{\mathbf{A}}_{R\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & = \mathbf{0} \quad [\psi_R = \mathbf{c}_S] \\
& \vec{\mathbf{y}}_S \geq \mathbf{0}, \; \overleftarrow{\mathbf{y}}_S \geq \mathbf{0} & & \\
& & \bar{\mathbf{A}}_{Z\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & = \mathbf{0} \quad [\psi_Z \; \textit{free}] \\
& & \vec{\mathbf{y}}_{\bar{S}} \geq \mathbf{0}, \; \overleftarrow{\mathbf{y}}_{\bar{S}} \geq \mathbf{0} & \\
& \vec{\mathbf{y}}_S \leq \vec{\mathbf{r}}_S, \; \overleftarrow{\mathbf{y}}_S \leq \overleftarrow{\mathbf{r}}_S & \vec{\mathbf{y}}_{\bar{S}} \leq \vec{\mathbf{r}}_{\bar{S}}, \; \overleftarrow{\mathbf{y}}_{\bar{S}} \leq \overleftarrow{\mathbf{r}}_{\bar{S}} &
\end{array}$$

Observe $\bar{\mathbf{c}}_S = \mathbf{0}$ (basic variables).

Master basis $\mathbf{I}_s$. DW subproblem in red.

Step 5 Find minimum reduced cost μ_S^k

Pricing of the variables

$S \subseteq B$, hence $\bar{\mathbf{c}}_S = \mathbf{0}$.
Therefore pricing of $\vec{\mathbf{y}}_{\bar{S}}$ and $\overleftarrow{\mathbf{y}}_{\bar{S}}$ needed to get partial direction $(\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k)$ followed by impact on $(\vec{\mathbf{y}}_S^k - \overleftarrow{\mathbf{y}}_S^k)$ to complete direction $\mathbf{y}^k = \begin{bmatrix} \mathbf{y}_S^k \\ \mathbf{y}_{\bar{S}}^k \end{bmatrix}$.

Primal/dual formulations of the pricing $\quad \psi_R^\intercal = \mathbf{c}_S^\intercal \mathbf{A}_{RS}^{-1}$

$$
\begin{aligned}
\max\ \mu \quad \text{st.}\quad \mu \mathbf{1}^\intercal &\leq \mathbf{c}_{\bar{S}}^\intercal - \psi_R^\intercal \mathbf{A}_{R\bar{S}} - \psi_Z^\intercal \bar{\mathbf{A}}_{Z\bar{S}} && [\vec{\mathbf{y}}_{\bar{S}}] \\
\mu \mathbf{1}^\intercal &\leq -(\mathbf{c}_{\bar{S}}^\intercal - \psi_R^\intercal \mathbf{A}_{R\bar{S}} - \psi_Z^\intercal \bar{\mathbf{A}}_{Z\bar{S}}) && [\overleftarrow{\mathbf{y}}_{\bar{S}}]
\end{aligned}
$$

A convex combination of the variables $\vec{\mathbf{y}}_{\bar{S}}$ and $\overleftarrow{\mathbf{y}}_{\bar{S}}$

$$
\begin{aligned}
\mu = \min\ & (\mathbf{c}_{\bar{S}}^\intercal - \psi_R^\intercal \mathbf{A}_{R\bar{S}})(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) \\
\text{st.}\ & \mathbf{1}^\intercal \vec{\mathbf{y}}_{\bar{S}} + \mathbf{1}^\intercal \overleftarrow{\mathbf{y}}_{\bar{S}} = 1 \quad [\mu] \\
& \mathbf{A}_{Z\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) = \mathbf{0} \quad [\psi_Z] \\
& \vec{\mathbf{y}}_{\bar{S}}, \overleftarrow{\mathbf{y}}_{\bar{S}} \geq \mathbf{0}, \overleftarrow{\mathbf{y}}_{\bar{S}_L} \leq \mathbf{0},\ \vec{\mathbf{y}}_{\bar{S}_U} \leq \mathbf{0}
\end{aligned}
$$

*** Optimal solution : μ_S^k, $\vec{\mathbf{y}}_{\bar{S}}^k$ and $\overleftarrow{\mathbf{y}}_{\bar{S}}^k$. ***

Step 6 Retrieve direction $\mathbf{y}_S^k$ of minimum reduced cost μ_S^k

… Complete direction $\mathbf{y}_S^k$

Given $\mu^k < 0$ and $(\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k)$, find impacts on $(\vec{\mathbf{y}}_S^k - \overleftarrow{\mathbf{y}}_S^k)$.

$$(\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) \quad + \quad \bar{\mathbf{A}}_{R\bar{S}}(\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k) \quad = \quad \mathbf{0}$$

$$\vec{\mathbf{y}}_S \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_S \geq \mathbf{0} \qquad\qquad \vec{y}_j \overleftarrow{y}_j = 0, \quad \forall j \in S_F$$

Direction $\mathbf{y}_S^k$

$$\mathbf{y}_S^k = \begin{bmatrix} (\vec{\mathbf{y}}_S^k - \overleftarrow{\mathbf{y}}_S^k) \\ (\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k) \end{bmatrix} = \begin{bmatrix} -\bar{\mathbf{A}}_{R\bar{S}}(\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k) \\ (\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k) \end{bmatrix} : \begin{bmatrix} \textit{Impact level} \quad (\textit{MP}) \\ \textit{Pricing level} \quad (\textit{SP}) \end{bmatrix}$$

$\vec{\mathbf{y}}_S^k, \overleftarrow{\mathbf{y}}_S^k$ *take positive and negative parts of* $-\bar{\mathbf{A}}_{R\bar{S}}(\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k)$, *respectively*.

Step 6 Compute maximum step-size ρ_S^k

$$\rho \begin{bmatrix} \vec{\mathbf{y}}_{S_F}^k \\ \vec{\mathbf{y}}_{S_L}^k \\ \vec{\mathbf{y}}_{S_U} \end{bmatrix} \leq \begin{bmatrix} \vec{\mathbf{r}}_{S_F}^k \\ \vec{\mathbf{r}}_{S_L}^k \\ \mathbf{0} \end{bmatrix}; \quad \rho \begin{bmatrix} \overleftarrow{\mathbf{y}}_{S_F}^k \\ \overleftarrow{\mathbf{y}}_{S_L}^k \\ \overleftarrow{\mathbf{y}}_{S_U}^k \end{bmatrix} \leq \begin{bmatrix} \overleftarrow{\mathbf{r}}_{S_F}^k \\ \mathbf{0} \\ \overleftarrow{\mathbf{r}}_{S_U}^k \end{bmatrix}$$

$$\rho \begin{bmatrix} \vec{\mathbf{y}}_{\bar{S}_F}^k \\ \vec{\mathbf{y}}_{\bar{S}_L}^k \\ \mathbf{0} \end{bmatrix} \leq \begin{bmatrix} \vec{\mathbf{r}}_{\bar{S}_F}^k \\ \vec{\mathbf{r}}_{\bar{S}_L}^k \\ \mathbf{0} \end{bmatrix}; \quad \rho \begin{bmatrix} \overleftarrow{\mathbf{y}}_{\bar{S}_F}^k \\ \mathbf{0} \\ \overleftarrow{\mathbf{y}}_{\bar{S}_U}^k \end{bmatrix} \leq \begin{bmatrix} \overleftarrow{\mathbf{r}}_{\bar{S}_F}^k \\ \mathbf{0} \\ \overleftarrow{\mathbf{r}}_{\bar{S}_U}^k \end{bmatrix}$$

10 out of 12 types of residual upper bonds to verify. $\emptyset \subseteq S \subseteq B$

Step 6 Compute maximum step-size ρ_S^k

$$\rho \begin{bmatrix} \vec{\mathbf{y}}_F^k \\ \vec{\mathbf{y}}_{B_L}^k \\ \vec{\mathbf{y}}_{B_U} \end{bmatrix} \leq \begin{bmatrix} \vec{\mathbf{r}}_F^k \\ \vec{\mathbf{r}}_{B_L}^k \\ \mathbf{0} \end{bmatrix}; \quad \rho \begin{bmatrix} \overleftarrow{\mathbf{y}}_F^k \\ \overleftarrow{\mathbf{y}}_{B_L}^k \\ \overleftarrow{\mathbf{y}}_{B_U}^k \end{bmatrix} \leq \begin{bmatrix} \overleftarrow{\mathbf{r}}_F^k \\ \mathbf{0} \\ \overleftarrow{\mathbf{r}}_{B_U}^k \end{bmatrix}$$

$$\rho \left[\vec{\mathbf{y}}_{N_L}^k \right] \leq \left[\vec{\mathbf{r}}_{N_L}^k \right]; \quad \rho \left[\overleftarrow{\mathbf{y}}_{N_U}^k \right] \leq \left[\overleftarrow{\mathbf{r}}_{N_U}^k \right]$$

8 types of residual upper bonds to verify for Primal Simplex. $S = B$

Step 6 Compute maximum step-size ρ_S^k

$$\rho \begin{bmatrix} \overleftarrow{\mathbf{y}}_P^k \\ \overleftarrow{\mathbf{y}}_Z^k \end{bmatrix} \leq \begin{bmatrix} \overleftarrow{\mathbf{r}}_P^k \\ \mathbf{0} \end{bmatrix}$$

Only 2 types of residual upper bonds to verify for PS in standard form.

$S = B = P \cup Z$ (positive and zero variables)

Step 6 Compute maximum step-size ρ_S^k

$$\rho \begin{bmatrix} \vec{\mathbf{y}}_F^k \\ \vec{\mathbf{y}}_L^k \end{bmatrix} \leq \begin{bmatrix} \vec{\mathbf{r}}_F^k \\ \vec{\mathbf{r}}_L^k \end{bmatrix}; \quad \rho \begin{bmatrix} \overleftarrow{\mathbf{y}}_F^k \\ \overleftarrow{\mathbf{y}}_U^k \end{bmatrix} \leq \begin{bmatrix} \overleftarrow{\mathbf{r}}_F^k \\ \overleftarrow{\mathbf{r}}_U^k \end{bmatrix}$$

4 types of <u>strictly positive</u> residual upper bonds to verify in MMCC. $S = \emptyset$

Step 6 Compute maximum step-size ρ_S^k

$$\rho\left[\ \vec{\mathbf{y}}_F^k\ \right] \leq \left[\ \vec{\mathbf{r}}_F^k\ \right];\quad \rho\left[\ \overleftarrow{\mathbf{y}}_F^k\ \right] \leq \left[\ \overleftarrow{\mathbf{r}}_F^k\ \right]$$

$$\rho\left[\ \vec{\mathbf{y}}_L^k\ \right] \leq \left[\ \vec{\mathbf{r}}_L^k\ \right];\quad \rho\left[\ \overleftarrow{\mathbf{y}}_U^k\ \right] \leq \left[\ \overleftarrow{\mathbf{r}}_U^k\ \right]$$

4 types of strictly positive residual upper bonds to verify in IPS. $S = F$

Special case #1 : $S = B$

$$\mathbf{T} = [\mathbf{A}_B \quad \emptyset], \quad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_B^{-1} \\ \emptyset \end{bmatrix}$$

$$B = F \cup B_L \cup B_U; \quad N = N_L \cup N_U$$

Special case #1 : $S = B$

$$\mathbf{T} = [\mathbf{A}_B \quad \emptyset], \quad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_B^{-1} \\ \emptyset \end{bmatrix}$$

$$B = F \cup B_L \cup B_U; \quad N = N_L \cup N_U$$

Primal simplex method (Dantzig 1945)

$$\begin{array}{llll}
\mathbf{c}^\intercal \mathbf{x}^k + \min \ \mathbf{c}_B^\intercal(\vec{\mathbf{y}}_B - \overleftarrow{\mathbf{y}}_B) & + & \mathbf{c}_{N_L}^\intercal \vec{\mathbf{y}}_{N_L} - \mathbf{c}_{N_U}^\intercal \overleftarrow{\mathbf{y}}_{N_U} & \\
\text{st.} \quad (\vec{\mathbf{y}}_B - \overleftarrow{\mathbf{y}}_B) & + & \bar{\mathbf{A}}_{N_L} \vec{\mathbf{y}}_{N_L} - \bar{\mathbf{A}}_{N_U} \overleftarrow{\mathbf{y}}_{N_U} & = \mathbf{0} \quad [\psi^\intercal = \mathbf{c}_B] \\
\vec{\mathbf{y}}_B \geq \mathbf{0}, \ \overleftarrow{\mathbf{y}}_B \geq \mathbf{0} & & & \\
& & \vec{\mathbf{y}}_{N_L} \geq \mathbf{0}, \ \overleftarrow{\mathbf{y}}_{N_U} \geq \mathbf{0} & \\
\vec{\mathbf{y}}_F \leq \vec{\mathbf{r}}_F, \ \overleftarrow{\mathbf{y}}_F \leq \overleftarrow{\mathbf{r}}_F & & & \\
\overleftarrow{\mathbf{y}}_{B_L} \leq \mathbf{0}, \ \vec{\mathbf{y}}_{B_U} \leq \mathbf{0} & & \vec{\mathbf{y}}_{N_L} \leq \vec{\mathbf{r}}_{N_L}, \overleftarrow{\mathbf{y}}_{N_U} \leq \overleftarrow{\mathbf{r}}_{N_U} & *\rho_B \geq 0*
\end{array}$$

Properties of PS

No equality constraints in the pricing problem.
Pricing contains • convex combination of the non-basic variables
• non-negativity restrictions (a cone).
Due to the step-size $*\rho_B \geq 0*$, possible degenerate pivots.
Oscillation of μ_B ; it may even not converge towards 0.

Special case #2 : $S = F \Rightarrow \bar{S} = L \cup U$

$$\mathbf{T} = \begin{bmatrix} \mathbf{A}_{RF} & \mathbf{0} \\ \mathbf{A}_{ZF} & \mathbf{I}_{m-r} \end{bmatrix}, \quad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_{RF}^{-1} & \mathbf{0} \\ -\mathbf{A}_{ZS}\mathbf{A}_{RF}^{-1} & \mathbf{I}_{m-r} \end{bmatrix}.$$

Special case #2 : $S = F \Rightarrow \bar{S} = L \cup U$

$$\mathbf{T} = \begin{bmatrix} \mathbf{A}_{RF} & \mathbf{0} \\ \mathbf{A}_{ZF} & \mathbf{I}_{m-r} \end{bmatrix}, \quad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_{RF}^{-1} & \mathbf{0} \\ -\mathbf{A}_{ZS}\mathbf{A}_{RF}^{-1} & \mathbf{I}_{m-r} \end{bmatrix}.$$

Improved Primal Simplex method (Elhallaoui et al. 2011)

$$\begin{array}{lrcrcll}
\mathbf{c}^{\intercal}\mathbf{x}^{k} + \min & \mathbf{c}_F^{\intercal}(\vec{\mathbf{y}}_F - \overleftarrow{\mathbf{y}}_F) & + & \mathbf{c}_L^{\intercal}\vec{\mathbf{y}}_L - \mathbf{c}_U^{\intercal}\overleftarrow{\mathbf{y}}_U & & & \\
\text{st.} & (\vec{\mathbf{y}}_F - \overleftarrow{\mathbf{y}}_F) & + & \bar{\mathbf{A}}_{RL}\vec{\mathbf{y}}_L - \bar{\mathbf{A}}_{RU}\overleftarrow{\mathbf{y}}_U & = & \mathbf{0} & [\psi_R = \mathbf{c}_F] \\
 & \vec{\mathbf{y}}_F \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_F \geq \mathbf{0} & & & & & \\
 & & & \bar{\mathbf{A}}_{ZL}\vec{\mathbf{y}}_L - \bar{\mathbf{A}}_{ZU}\overleftarrow{\mathbf{y}}_U & = & \mathbf{0} & [\psi_Z] \\
 & & & \vec{\mathbf{y}}_L \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_U \geq \mathbf{0} & & & \\
 & \vec{\mathbf{y}}_F \leq \vec{\mathbf{r}}_F,\ \overleftarrow{\mathbf{y}}_F \leq \overleftarrow{\mathbf{r}}_F, & & \vec{\mathbf{y}}_L \leq \vec{\mathbf{r}}_L, \overleftarrow{\mathbf{y}}_U \leq \overleftarrow{\mathbf{r}}_U & & & *\rho > 0*
\end{array}$$

Properties of IPS

f equality constraints in the master problem, $m - f$ in the pricing problem.

Non-degenerate pivots only ($\rho_F > 0$).

$z^0 > z^1 > z^2 > \cdots = z^{\star}$ cost **strictly decreasing** at each iteration ($\rho_F > 0$).

Oscillation of μ_F but converging towards 0.

Special case #3 : $S = \emptyset$

Select $S = \emptyset$. $\mathbf{T} = \begin{bmatrix} \emptyset & \mathbf{I}_m \end{bmatrix}, \mathbf{T}^{-1} = \begin{bmatrix} \emptyset \\ \mathbf{I}_m \end{bmatrix}$

Special case #3 : $S = \emptyset$

Select $S = \emptyset$. $\mathbf{T} = \begin{bmatrix} \emptyset & \mathbf{I}_m \end{bmatrix}$, $\mathbf{T}^{-1} = \begin{bmatrix} \emptyset \\ \mathbf{I}_m \end{bmatrix}$

Minimum mean cycle-canceling algorithm adapted for LP

$$
\begin{array}{llrll}
\mathbf{c}^\intercal \mathbf{x}^k + & \min & \mathbf{c}^\intercal(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) & & \\
 & \text{st.} & \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) & = \mathbf{0} & \textit{Directions} \\
 & & \vec{\mathbf{y}},\ \overleftarrow{\mathbf{y}} & \geq \mathbf{0} & \textit{in the cone} \\
 & & \overleftarrow{\mathbf{y}}_L,\ \vec{\mathbf{y}}_U & \leq \mathbf{0} & \textit{at vertex } \mathbf{x}^k \\
 & & \vec{\mathbf{y}}_F, \overleftarrow{\mathbf{y}}_F, \vec{\mathbf{y}}_L, \overleftarrow{\mathbf{y}}_U & \leq \vec{\mathbf{r}}_F^k, \overleftarrow{\mathbf{r}}_F^k, \vec{\mathbf{r}}_L^k, \overleftarrow{\mathbf{r}}_U^k & *\textit{Step size } \rho_\emptyset > 0*
\end{array}
$$

Properties of MMCC

All equality constraints in the pricing problem.

Upper bounds in the master problem.

$z^0 > z^1 > z^2 > \cdots = z^\star$ cost **strictly decreasing** at each iteration ($\rho_\emptyset > 0$).

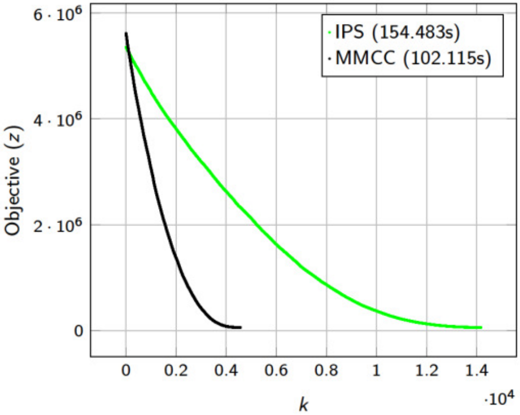
$\mu^0 \leq \mu^1 \leq \mu^2 \leq \cdots = 0$ smallest reduced cost **non decreasing**.

MMCC is strongly polynomial for network flow problems in $O(mn)$ phases.

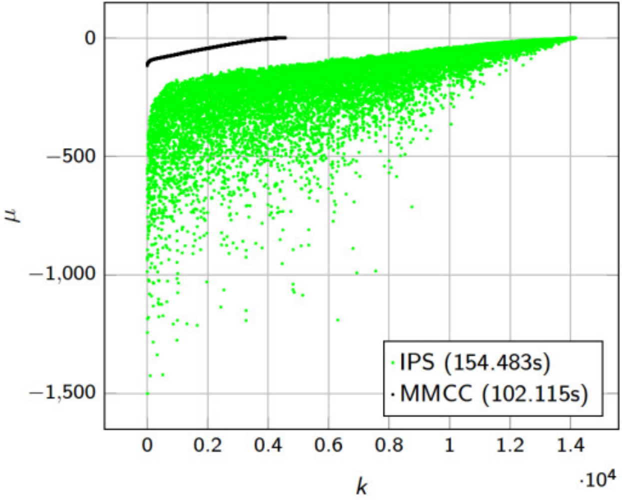
Goldberg and Tarjan (1989), Radzick and Goldberg (1994)

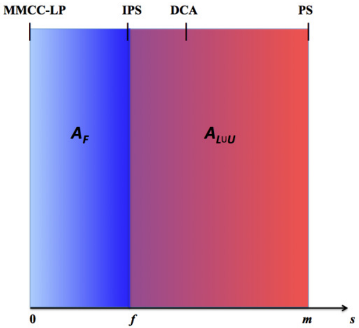
Illustration of z (Network with n=1025, m=91,220)

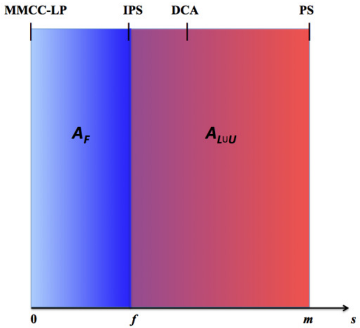
Computational results



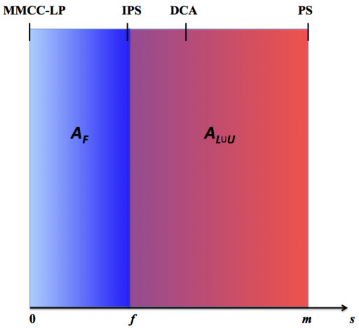
COMPUTATIONAL RESULTS



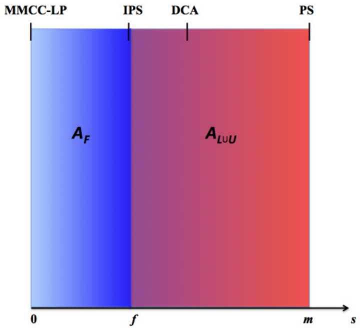

MMCC-LP
IPS
DCA
PS
A_F
$A_{L\cup U}$
0
f
m
s



- $S \cap \{L \cup U\} \neq \emptyset$: It may come up with degenerate pivots and not converge.
  - Primal Simplex method (PS). $S = B, * \rho_B \geq 0 *$



- $S \cap \{L \cup U\} \neq \emptyset$: It may come up with degenerate pivots and not converge.
  - Primal Simplex method (PS). $S = B$, $* \ \rho_B \geq 0 \ *$
- $\emptyset \subseteq S \subseteq F$: It ensures a non-degenerate pivot at every iteration.
  - Improved Primal Simplex algorithm (IPS). $S = F$, $* \ \rho_F > 0 \ *$



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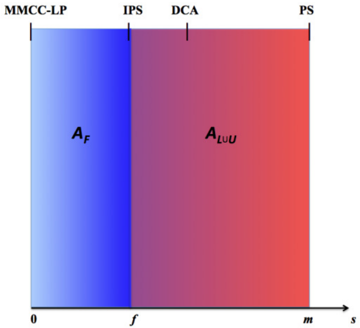
- $\emptyset \subseteq S \subseteq F$: It ensures a non-degenerate pivot at every iteration.
  - Improved Primal Simplex algorithm (IPS). $S = F$, $*\ \rho_F > 0\ *$
  - Minimum mean cycle-canceling algorithm (MMCC) $S = \emptyset$, $*\ \rho_\emptyset > 0\ *$



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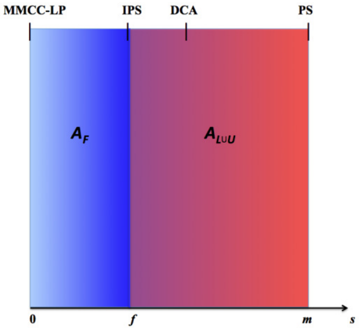
  - Minimum mean cycle-canceling algorithm (MMCC) $S = \emptyset$, $* \ \rho_\emptyset > 0 \ *$
    Strongly polynomial for network flow problems.



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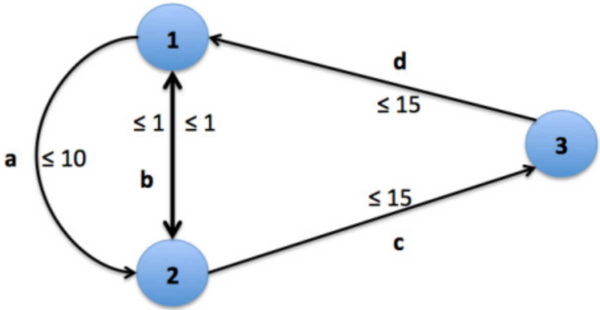
  - Minimum mean cycle-canceling algorithm (MMCC) $S = \emptyset$, $* \ \rho_\emptyset > 0 \ *$

    Strongly polynomial for network flow problems.

  * $S \subset F$: Optimal direction $\mathbf{y}_S^k$ can be an interior ray.

MMCC:

non-extreme ray a-c-d



*** Imagine the same transformation is kept for a while...

- $\mathbf{T} := [\, \mathbf{\Lambda},\ \mathbf{\Lambda}^{\perp}]$, where $\mathbf{\Lambda}$ is a set of independent columns.
- $\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_R \\ \mathbf{\Lambda}_Z \end{bmatrix}$, where $\mathbf{\Lambda}_R$ is a set of s independent rows of $\mathbf{\Lambda}$.
- $\mathbf{T} = \begin{bmatrix} \mathbf{\Lambda}_R & \mathbf{0} \\ \mathbf{\Lambda}_Z & \mathbf{I}_{m-s} \end{bmatrix}$ $\qquad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{\Lambda}_R^{-1} & \mathbf{0} \\ \mathbf{\Lambda}_Z \mathbf{\Lambda}_R^{-1} & \mathbf{I}_{m-s} \end{bmatrix}$
- $\mathbf{T}^{-1}$ splits row-space $\mathbb{R}^m$ of $LP(\mathbf{x}^k)$ into two vector subspaces $\mathbf{V}$ and $\mathbf{V}^{\perp}$.
- Vector subspace basis $\mathbf{\Lambda}$ spans $\mathbf{V}$ of dimension $0 \leq s \leq m$.
- Vector $\mathbf{a} \in \mathbf{V}$ if and only if $\bar{\mathbf{a}}_Z = \mathbf{0}$, where $\bar{\mathbf{a}} = \mathbf{T}^{-1}\mathbf{a} = \begin{bmatrix} \bar{\mathbf{a}}_R \\ \mathbf{0} \end{bmatrix}$.
- $S \subseteq B$: index set of basic columns spanned by $\mathbf{\Lambda}$.
- Algorithmic properties derived according to subset S.

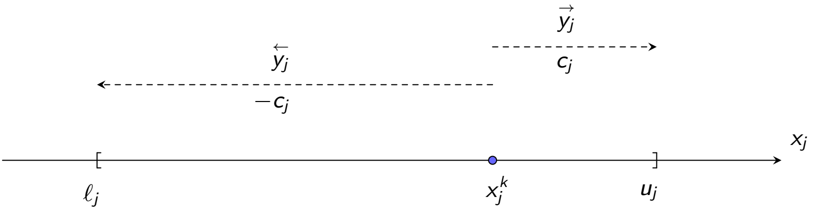
DYNAMIC CONSTRAINT AGGREGATION FOR SET PARTITIONING (ELHALLAOUI ET AL. 2005)

The partition of the row-set is derived from the f groups of identical rows of A_F.

$$\overset{\mathbf{A}_F}{\begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ 1 & & 1 & \\ 1 & & & 1 \\ 1 & 1 & & \\ & 1 & 1 & \\ 1 & & & 1 \end{pmatrix}} \mapsto \overset{\boldsymbol{\Lambda}}{\left(\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ \hline 1 & & & & \\ & 1 & & & \\ & & & & 1 \end{array}\right)}.$$

PROPERTIES

$F \subseteq S \subseteq B$ for fractional solutions but $S = F$ (or $S = \emptyset$) for binary solutions.
Degenerate pivots may occur, $\rho \geq 0$, if $r > f$.



RESIDUAL PROBLEM $LP(\mathbf{x}^k)$

$$\begin{aligned} z^\star := \mathbf{c}^\mathsf{T}\mathbf{x}^k + \min \quad & \mathbf{c}^\mathsf{T}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) \\ \text{st.} \quad & \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \mathbf{0} \\ & \mathbf{0} \leq \vec{\mathbf{y}} \leq \vec{\mathbf{r}}^k \\ & \mathbf{0} \leq \overleftarrow{\mathbf{y}} \leq \overleftarrow{\mathbf{r}}^k \end{aligned}$$