Improved Branch-Cut-and-Price for Capacitated Vehicle Routing

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The Capacitated Vehicle Routing Problem (CVRP)

Given:

- An undirected graph $G = (V, E)$ with $V = \{0, \ldots, n\}$
  - Vertex 0 is the *depot*, $V_+ = \{1, \ldots, n\}$ is the set of *customers*
  - Each edge $e \in E$ has a *cost* c_e.
- Customer demands are $d_1, \ldots, d_n$
- K *vehicles* with *capacity* Q

Solution:

- A set of K routes where:
  - each route starts and ends at the depot
  - each customer is visited by exactly one vehicle
  - the total demand of customers visited in a route is at most Q

- The most classical VRP variant, proposed by Dantzig and Ramser in 1959.

Two Index Formulation

Nobert, Laporte (1983)

- $G = (V, E)$ complete graph with $V = \{0, \ldots, n\}$, E as edge set
- $V_+ = \{1, \ldots, n\}$ are *customers*, vertex 0 is the depot
- nonnegative *travel costs*, c_{ij}, $e = (i, j) \in E$
- $d(i)$ customer's i demand $i \in V_+$, C is the capacity of the K vehicles
- for $S \subseteq V$, let $d(S) = \sum_{i \in S} d(i)$, and $k(S) = \lceil d(S)/Q \rceil$

$$
\begin{array}{llll}
\text{(TIF)} \quad \min & \sum_{e \in E} c_e x_e & & (1) \\
\quad \textit{subject to} & & & (2) \\
& \sum_{e \in \delta(i)} x_e = 2, & \forall\ i \in V_+ & (3) \\
& \sum_{e \in \delta(0)} x_e = 2K, & & (4) \\
& \sum_{e \in \delta(S)} x_e \geq 2k(S), & \forall\ S \subseteq V_+, & (5) \\
& x_e \in \{0, 1\}, & \forall\ e \notin \delta(0), & (6) \\
& x_e \in \{0, 1, 2\}, & \forall\ e \in \delta(0). & (7)
\end{array}
$$

Arc Load Formulation

- $G_Q = (V, A_Q)$
- A_Q contains arcs $(i,j)^q$, for all $i \in V_+, j \in V$ and for all $q = d_i, \ldots, Q$
- $c_a = c_{ij} = c_{ji}$, $e = (i,j) \in E$
- x^q_{ij} indicates that some vehicle goes from i to j carrying load q
- The Arc-Load indexed Formulation is:

$$\text{(ALF)} \quad \min \sum_{a^q \in A_Q} c_a x^q_a \tag{8}$$

$$\textit{subject to} \tag{9}$$

$$\sum_{a^q \in \delta^+(\{i\})} x^q_a = 1, \qquad \forall i \in V_+, \tag{10}$$

$$\sum_{a^q \in \delta^+(\{0\})} x^q_a = K, \tag{11}$$

$$\sum_{a^{q-d_i} \in \delta^-(\{i\})} x^{q-d_i}_a - \sum_{a^q \in \delta^+(\{i\})} x^q_a = 0, \qquad \forall i \in V_+, q = d_i, \ldots, \tag{12}$$

$$x^q_a \geq 0, \qquad \forall a^q \in A_Q, \tag{13}$$

$$x \text{ integer.} \tag{14}$$

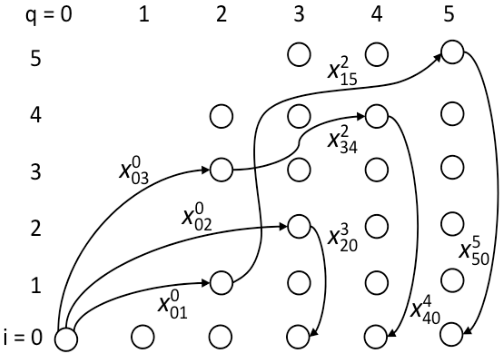


Figure: Representation of a solution as a set of paths in $\mathcal{N}$

Set Partitioning Formulation

$$\sum_{r\in\Omega} a^{ij}_{rq}\lambda_r = x^q_{ij}, \quad \forall (i,j)^q \in A_Q. \tag{15}$$

Substituting the x variables and relaxing the integrality, the Dantzig-Wolfe Master LP is written as:

$$\text{(DWM)} \quad \min \quad \sum_{r\in\Omega}\left(\sum_{(i,j)^q\in A_Q} a^{ij}_{rq} c_{ij}\right)\lambda_r \tag{16}$$

S.t.

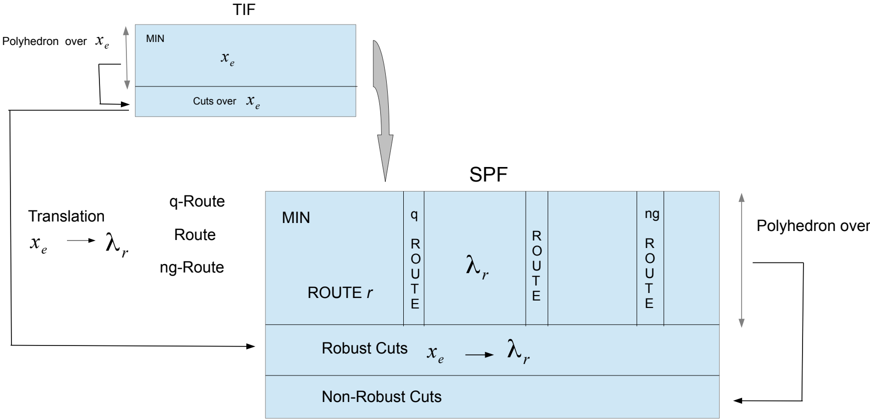
$$\sum_{r\in\Omega}\left(\sum_{(i,j)^q\in\delta^+(\{i\})} a^{ij}_{rq}\right)\lambda_r = 1, \qquad \forall i \in V_+, \tag{17}$$

$$\sum_{r\in\Omega}\left(\sum_{(i,j)^q\in\delta^+(\{0\})} a^{ij}_{rq}\right)\lambda_r = K, \tag{18}$$

$$\lambda_r \geq 0 \qquad \forall r \in \Omega. \tag{19}$$

A generic constraint l of format $\sum_{(i,j)^q\in A_Q} \alpha^{lq}_{ij} x^q_{ij} \geq b_l$ can also be included in the DWM, using the variable substitution (15)

Optimizing over both TIF and SPF ... and ALF



Families of cuts over edge variables x_e (Robust)

- Robust: No change on the pricing problem. Only Dual Values change.

- rounded capacity (RCC in TIF)
- framed capacities
- strengthened combs
- multistars
- extended hypotours

Package CVRPSEP by J. Lysgaard provide effective heuristic separation procedures.

- Only RCC and strengthened combs used in this BCP

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Subset Row Cuts (SRCs)

a_i^r is the number of times that customer i appears in route r.

Given $C \subseteq V_+$ and a multiplier p, the (C, p)-Subset Row Cut is:

$$\sum_{r \in \Omega} \left\lfloor p \sum_{i \in C} a_i^r \right\rfloor \lambda_r \leq \lfloor p|C| \rfloor \quad (20)$$

Non-robust cut obtained by a Chvátal-Gomory rounding of $|C|$ constraints in the SPF (Jepsen et al. [2008]):

$$\sum_{r \in \Omega} a_i^r \lambda_r = 1, \quad \forall i \in C \quad (21)$$

3SRCs

The cuts where $|C| = 3$ and $p = 1/2$ are called **3-Subset Row Cuts (3SRCs)**

- Used in Baldacci et al. [2011] and Contardo [2012].
- $\lfloor p|C| \rfloor = 1$
- If $\sum_{i \in C} a_i^r$ is equal to 2 or 3, then $\left\lfloor p \sum_{i \in C} a_i^r \right\rfloor = 1.$
- If $\sum_{i \in C} a_i^r$ is equal to 4 or 5, then $\left\lfloor p \sum_{i \in C} a_i^r \right\rfloor = 2.$
- $\ldots$

In a feasible solution, at most one route can enter in C twice or thrice.

Interesting SRCs

$|C| = 1$ and $p = 1/2$, **1-Subset Row Cuts (1SRCs)**:

- Forbid routes that revisit a certain vertex
- Similar cut used in Contardo [2012]

$|C| = 4$ and $p = 2/3$, **4SRCs**
$|C| = 5$ and $p = 1/3$, **5,1SRCs**
$|C| = 5$ and $p = 1/2$, **5,2SRCs**

SRCs versus Pricing

- SRCs demonstrated to be effective *boundwise*.
- Non-robust cuts definitely impacts the pricing.
- How to balance pricing and SRCs effectiveness?
  - Choose C with clients close to each other?
  - Remember only C?

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limited memory Subset Row Cuts (lm-SRCs)

Given $C \subseteq V_+$, a memory set M, $C \subseteq M \subseteq V_+$, and a multiplier p, the limited memory (C, M, p)-Subset Row Cut is:

$$\sum_{r \in \Omega} \alpha(C, M, p, r)\lambda_r \leq \lfloor p|C| \rfloor \,, \tag{22}$$

where the coefficient of a route r is computed as:

function $\alpha(C, M, p, r)$
$coeff \leftarrow 0$, $state \leftarrow 0$
for every vertex $i \in r$ (in order) **do**
    if $i \notin M$ **then**
        $state \leftarrow 0$
    else if $i \in C$ **then**
        $state \leftarrow state + p$
        if $state \geq 1$ **then**
            $coeff \leftarrow coeff + 1$, $state \leftarrow state - 1$
return $coeff$

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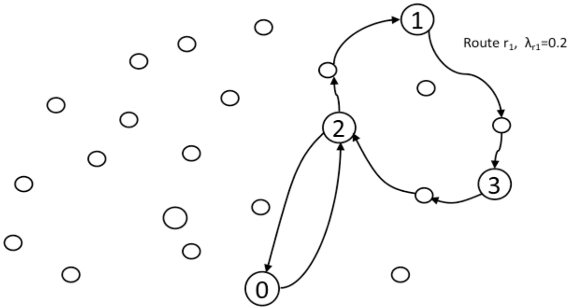
    else if $i \in C$ **then**
      $state \leftarrow state + p$
      if $state \geq 1$ **then**
        $coeff \leftarrow coeff + 1$, $state \leftarrow state - 1$
return $coeff$

- If $M = V_+$, the function returns $\lfloor p \sum_{i \in C} a_i^r \rfloor$
- Otherwise, the lm-SRC may be a weakening of the corresponding SRC

Separation of lm-SRCs



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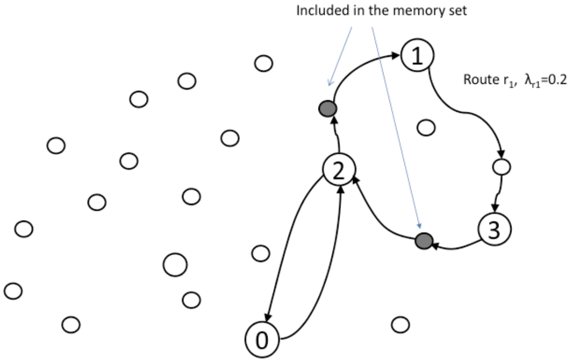

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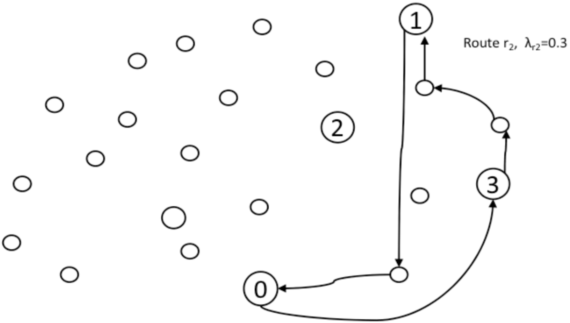

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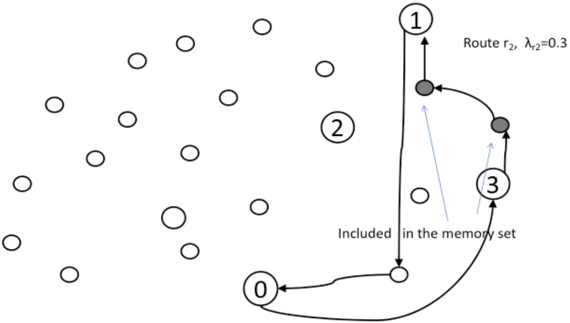

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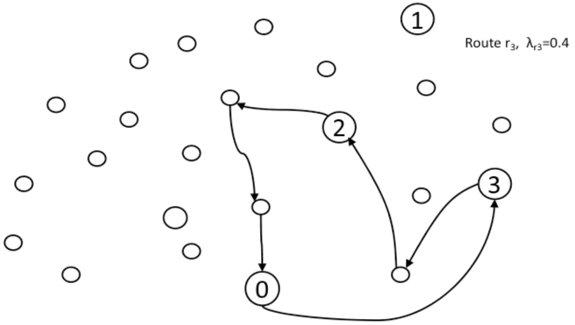

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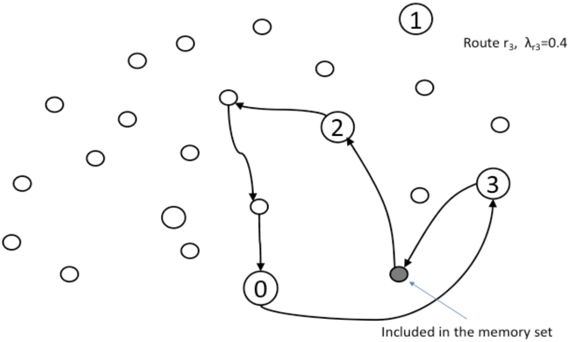


Suppose that nS lm-SRCs are in the SPF, cut s has dual variable σ_s. Those variables **penalize** the reduced cost of some paths. $S(P)$ is the vector of states a partial path P, calculated by the α function.

- Given labels $L(P_1)$ and $L(P_2)$, when $S(P_1)[s] > S(P_2)[s]$, it is possible that a completion for P_1 is penalized by σ_s but the same completion for P_2 is not.

Definition

A label $L(P_1)$ dominates another label $L(P_2)$ in the same bucket if $\bar{c}(P_1) \leq \bar{c}(P_2) + \sum_{1 \leq s \leq n_S : S(P_1)[s] > S(P_2)[s]} \sigma_s$ and every valid completion for P_2 is also a valid completion for P_1.

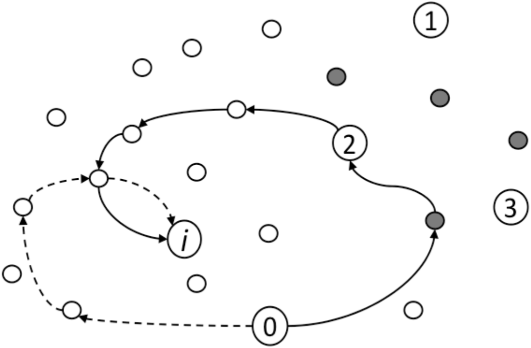


Figure: Solid path may only dominate the dashed path because the 3-SRC $\{1, 2, 3\}$ is already forgotten at $F(i, q)$.

Fixing by Reduced Cost - on ALF

Run Forward Labeling
Run Backward Labeling
for all $(i,j)^q \in A_Q$ such that $x^q_{i,j}$ is not fixed to 0 **do**
  for all $L_1 = (\bar{c}_1, i, q, \Pi_1, S_1, \_) \in F(i,q)$ in RC order **do**
    for all $L_2 = (\bar{c}_2, j, q + d_i, \Pi_2, S_2, \_) \in B(j, q + d_i)$ in RC order **do**
      $c_r \leftarrow c_1 + \bar{c}^q_{ij} + c_2$
      if $c_r > gap$ **then break**
      if $\Pi_1 \cap \Pi_2 = \emptyset$ **then**
        for $s := 1, \ldots, n_S$ **do**
          if $S_1[s] + S_2[s] \geq 1$ **then**
            $c_r \leftarrow c_r - \sigma_s$
        if $c_r <= gap$ **then** Fix variable $x^q_{i,j}$ to 0

Fixing by Reduced Cost - on ALF

- 95% of ALF variable are fixed
- Helps keeping the pricing under control

Algorithm Summary: Cuts

- Robust cuts
  - Rounded Capacity
  - Strengthened Comb
- Non-robust cuts
  - lm-SRC
- Post-enumeration cuts
  - SRC
  - Clique

Algorithm Summary: Pricing

- The label setting dynamic programming algorithm handles:
  - ng-routes ($ng = 8$)
  - lm-SRCs (1SRCs, 3SRCs, 4SRCs, 5SRCs)
- Features:
  - **Bidirectional Search**
  - **Completion Bounds** - Lower bounds on the cost of all possible extensions of a label, used to discard those that can not be optimal.
  - Fast and effective heuristics. Exact pricing called a few times per node (Bucket Pruning).

The most critical part of the BCP.

Algorithm Summary: Non-robustness control

Even with all the care in their separation, lm-SRCs are indeed "non-robust":

- The pricing may be handling several hundreds lm-SRCs efficiently. Then, in some node of the tree, it suddenly becomes 100 or even 1000 times slower!
- In those cases it is necessary to **roll back**, removing the offending cuts. The node lower bound decreases, but the BCP does not halt.
- Base # of labels: RL
- Delete the latest lm-SRC's added if # of labels reach 5 RL

Algorithm Summary: Miscellanea

- Variable Fixing
- Strong Branching:
  - Hierarchical, 3 levels
  - Uses history of past branchings
  - Aggressive, up to 300 candidates can be tested
  - Uses estimates of the subtree size determining the SB effort in each node
- Enumeration:
  - Performed when the node gap is sufficiently small for generating a pool with less than 20M routes
  - Ordinary branching occurs in enumerated nodes
  - The MIP solver only finishes the node when the pool has less than 30K routes
- Branch-and-cut: In the root node, when column generation is having severe convergence problems, it may try BC

Overall Results

		BMR11			Con12			Rop12		
Class	NP	Opt	Gap	Time	Opt	Gap	Time	Opt	Gap	Time
A	22	22	0.13	30	22	0.07	59	22	0.57	53
B	20	20	0.06	67	20	0.05	89	20	0.25	208
E-M	12	9	0.49	303	10	0.30	2807	10	0.96	44295
F	3	2	0.11	164	2	0.06	3	3	0.25	2163
P	24	24	0.23	85	24	0.13	43	24	0.69	280
Total	81	77			78			79		
Machine		Xeon X7350 2.93GHz			Xeon E5462 2.8GHz			Core i7-2620M 2.7GHz		

		This BCP		
Class	NP	Opt	Gap	Time
A	22	22	0.03	5.6
B	20	20	0.04	6.2
E-M	12	12	0.19	3669
F	3	3	0.00	3679
P	24	24	0.07	32.7
Total	81	81		
Machine		Core i7-3770 3.4GHz		

Detailed Results: M-n151-k12

Algo	Machine	Root LB	Final LB	Total Time
BMR11	X7350 2.93GHz	1004.3	1004.3	380
Contardo12	E5462 2.8GHz	1012.5	**1015**	19699
Ropke12	i7-2620M 2.7GHz	1001.5	**1015**	417146
This BCP	i7-3770 3.4GHz	1012.8	**1015**	212
Basic BCP	i7-3770 3.4GHz	1001.5	**1015**	7958

Basic BCP: stripped down version with only the cuts used in Ropke12, without SRCs or enumeration. Illustrates the gains that can be obtained only with "implementation details".

Detailed Results: M-n200-k17

Algo	Machine	Root LB	Final LB	Total Time
BMR11	X7350 2.93GHz	1258.7	1258.7	436
Contardo12	E5462 2.8GHz	1265.1	1265.1	34350
Ropke12	i7-2620M 2.7GHz	1255.3	1261.4	7200
This BCP	i7-3770 3.4GHz	1268.7	**1275**	3581

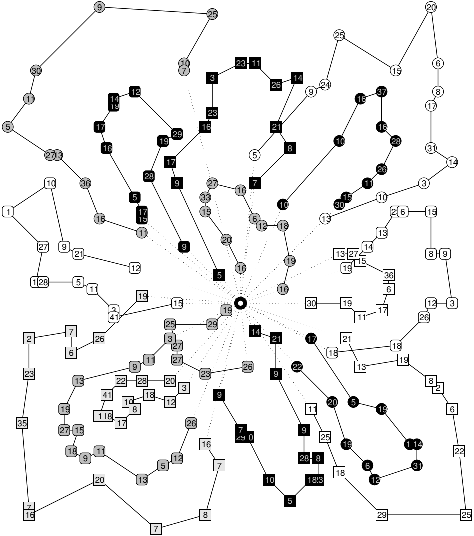
The Basic BCP can not solve this instance. The Im-SRCs are fundamental.

Detailed Results: M-n200-k16

Algo	Machine	Root LB	Final LB	Total Time
BMR11	X7350 2.93GHz	1256.6	1256.6	319
Contardo12	E5462 2.8GHz	1263.0	1263.0	265588
Ropke12	i7-2620M 2.7GHz	1253.0	1258.2	7200
This BCP	i7-3770 3.4GHz	1266.5	**1274**	39869

Previous upper bound: 1278.

Optimal solution of M-n200-k16 ($Q = 200$), cost 1274



M-n200 with fractional costs (C5)

In the heuristics literature it is usual not to round the edge costs and not to fix the number of vehicles.

Algorithm	Best Sol
Rochat and Taillard [1995]	1291.45
Pisinger and Røpke [2007]	1297.12
Mester and Bräysy [2007]	**1291.29**
Nagata and Bräysy [2009]	1291.45
Vidal et al. [2012]	1291.45
Subramanian et al. [2012]	1291.45

The optimal value is indeed 1291.29, proved in 18690 seconds.

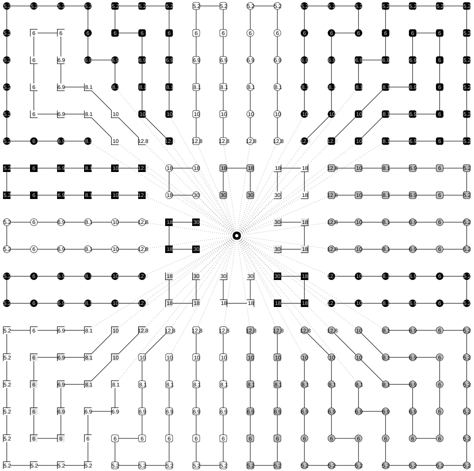
Golden Instances

Golden, Wasil, Kelly, and Chao [1998] proposed 12 instances, ranging from 240 to 483 customers.

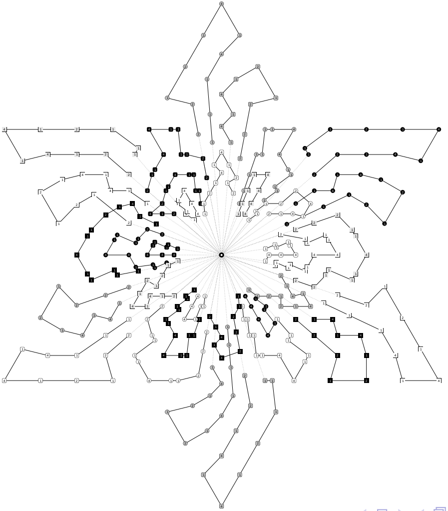
- Appear frequently in the literature on heuristic methods
- Until now, considered to be far beyond the reach of exact methods

Four instances could be solved, those with 240, 300, 320, and 360 customers.

Optimal solution of Golden_14 (320 customers, $Q = 1000$), cost 1080.55, 30 routes



Optimal solution of Golden_19 (360 customers, $Q = 20$), cost 1365.60, 33 routes



Comparison of algorithms over hard instances

Ins	Q	Alg	UB	RLB1	ER1	RLB2	ER2	RT(s)	FLB	Nodes	TT(s)
M-n151-k12	200	BMR11	1015	1004.3	-			380	1004.3	1	380
		Con12	1015	1008.9	4.0M	1012.5	13K	19041	**1015**	1	19699
		Rop12	1015	1001.5					**1015**	5268	417146
		BCP	1015	1011.7	59K	1012.8	8K	178	**1015**	1	212
		BBCP	1015	1001.5				109	**1015**	2537	7958
M-n200-k16	200	BMR11		1256.6	-			319	1256.6	1	319
		Con12	1278	1263.0	-	-	-	265589	1263.0	1	265589
		Rop12	1278	1253.0					1258.2	106	7200
		BCP	1278	1266.5	-	-	-	956	**1274**	97	39869
M-n200-k17	200	BMR11	1275	1258.7	-			436	1258.7	1	436
		Con12	1275	1265.1	-	-	-	34351	1265.1	1	34351
		Rop12	1276	1255.3					1261.4	144	7200
		BCP	1275	1268.7	-	-	-	537	**1275**	15	3581
C4 (150,12)	200	BCP	1028.42	1025.1	500k	1026.25	62k	430	**1028.42**	3	783
C5 (199,16)	200	BCP	1291.29	1284.14	-	-	-	595	**1291.29**	59	18690
G17 (240, 22)	20	BCP	707.76	705.54	-	-	-	1010	**707.76**	13	25203
G13 (252,-)	1000	BCP	857.19	851.97	-	-	-	21749	851.97	1	21749
G9 (255,-)	1000	BCP	579.71	576.88	-	-	-	9363	576.88	1	9363
G18 (300,27)	20	BCP	995.13	993.42	-	-	-	1030	**995.13**	15	25690
G14 (320,30)	1000	BCP	1080.55	1076.03	-	-	-	6330	**1080.55**	$\approx$ 3K	$\approx$ 36 days
G10 (323,-)	1000	BCP	736.26	731.13	-	-	-	16021	731.13	1	16021
G19 (360, 33)	20	BCP	1365.60	1363.10	-	-	-	1264	**1365.60**	239	260345

Conclusions

- Im-SRC's is a weakening of the SRC's biased by the pricing.
- ALF still holds a lot of unexplored information
- Formulations and polyhedral cuts over extended formulations may be treated without the use of a heavy LP.
- Impact on other variants of VRP ? (Rich and Poor)

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Sharing

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- Eduardo's saying:

  Before going to sleep, ask what you have done for the CVRP during the day,

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TAK!

Publications

- Martinelli, R.; Pecin, D.; Poggi, M. – **Efficient Elementary and Restricted Non-elementary Route Pricing for Routing Problems**. European Journal of Operational Research (EJOR), Online, 2014.
- Diego, P.; Pessoa, A.; Poggi, M.; Uchoa, E. – **Improved Branch-Cut-and-Price for Capacitated Vehicle Routing**. Integer Programming and Combinatorial Optimization (IPCO), June 2014 (full paper on the way).