Min-Max vs. Min-Sum Vehicle Routing: A Worst-Case Analysis

L. Bertazzi, B. Golden, and X. Wang

Route 2014
Denmark
June 2014

Introduction

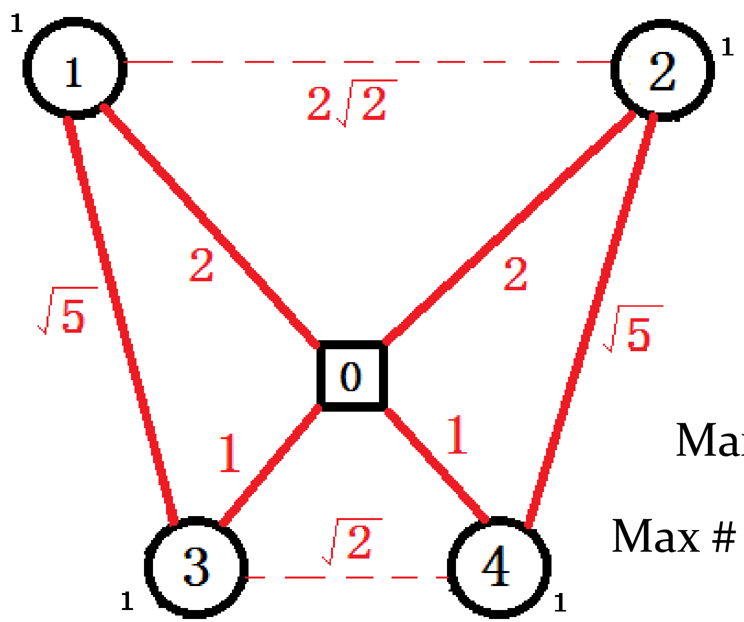
- In the min-sum VRP, the objective is to minimize the total cost incurred over all the routes
- In the min-max VRP, the objective is to minimize the maximum cost incurred by any one of the routes
- Suppose we have computer code that solves the min-sum VRP, how poorly can it do on the min-max VRP?
- Suppose we have computer code that solves the min-max VRP, how poorly can it do on the min-sum VRP?

Introduction

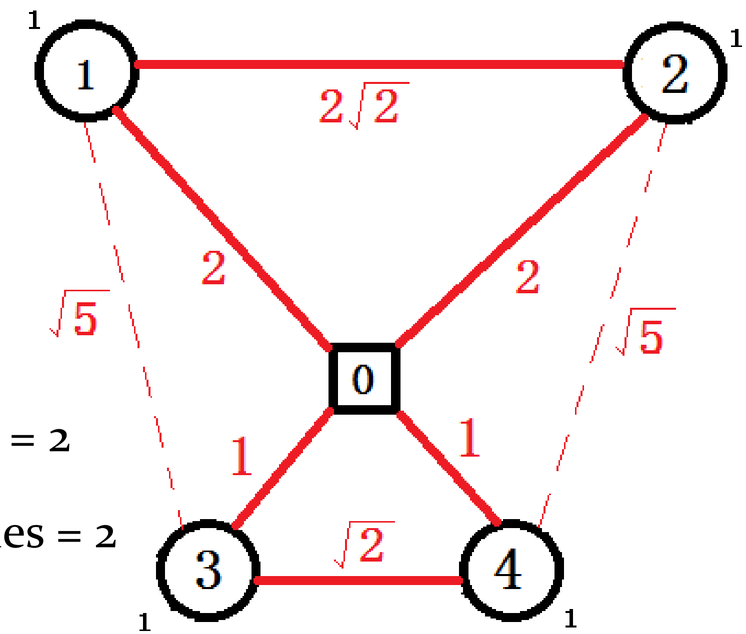
- Applications of the min-max objective
  - Disaster relief efforts
    - Serve all victims as soon as possible
  - Computer networks
    - Minimize maximum latency between a server and a client
  - Workload balance
    - Balance amount of work among drivers and/or across a time horizon

An Instance of the VRP

The min-max solution



The min-sum solution



Max load = 2

Max # vehicles = 2

The min-max solution

total cost $= 6 + 2\sqrt{5} \approx 10.47$

min-max cost $= 3 + \sqrt{5} \approx 5.24$

The min-sum solution

total cost $= 6 + 3\sqrt{2} \approx 10.24$

min-max cost $= 4 + 2\sqrt{2} \approx 6.83$

Motivation behind our Worst-Case Study

- Observation: The min-max solution has a slightly higher (2.2%) total cost, but it has a much smaller (23.3%) min-max cost
  - Also, the routes are better balanced
- Is this always the case?
- What is the worst-case ratio of the cost of the longest route in the min-sum VRP to the cost of the longest route in the min-max VRP?
- What is the worst-case ratio of the total cost of the min-max VRP to the total cost of the min-sum VRP?

Variants of the VRP Studied

- Capacitated VRP with infinitely many vehicles (CVRP_INF)
- Capacitated VRP with a finite number of vehicles (CVRP_k)
- Multiple TSP (MTSP_k)
- Service time VRP with a finite number of vehicles (SVRP_k)

CVRP_INF

- Capacitated VRP with an infinite number of vehicles
  - r^{∞}_{MM} : the cost of the longest route of the optimal min-max solution
  - r^{∞}_{MS} : the cost of the longest route of the optimal min-sum solution
  - z^{∞}_{MM} : the total cost of the optimal min-max solution
  - z^{∞}_{MS} : the total cost of the optimal min-sum solution
  - The superscript denotes the variant

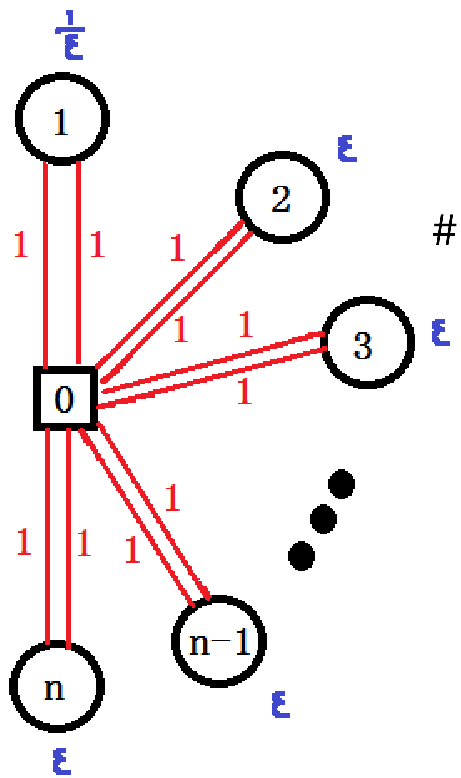
A Preview of Things to Come

- For each variant, we present worst-case bounds

- In addition, we show instances that demonstrate that the worst-case bounds are tight

CVRP_INF

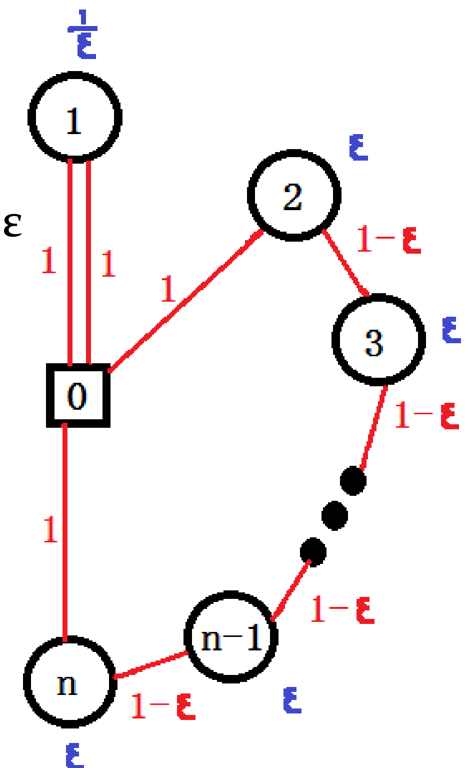
$$\boxed{r_{MS}^{\infty} / r_{MM}^{\infty} \to \infty}$$



$$r_{MM}^{\infty} = 2$$

customers: n =1+1/ε

capacity = 1/ε

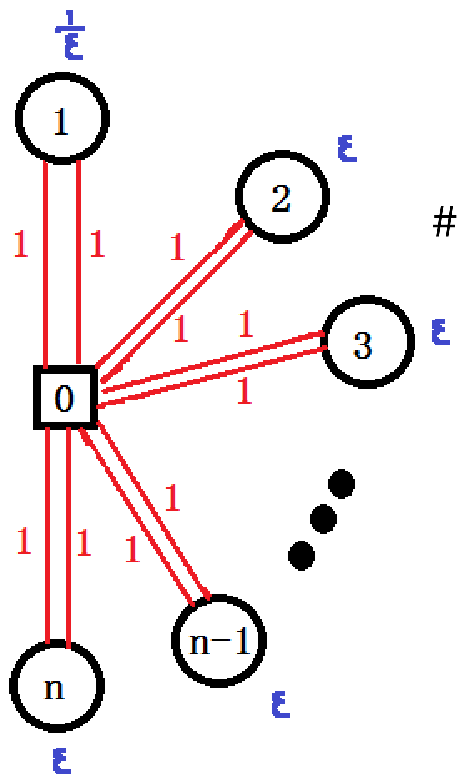


$$r_{MS}^{\infty} = 2 + (n-2)(1-\varepsilon)$$

$$= 1/\varepsilon + \varepsilon$$

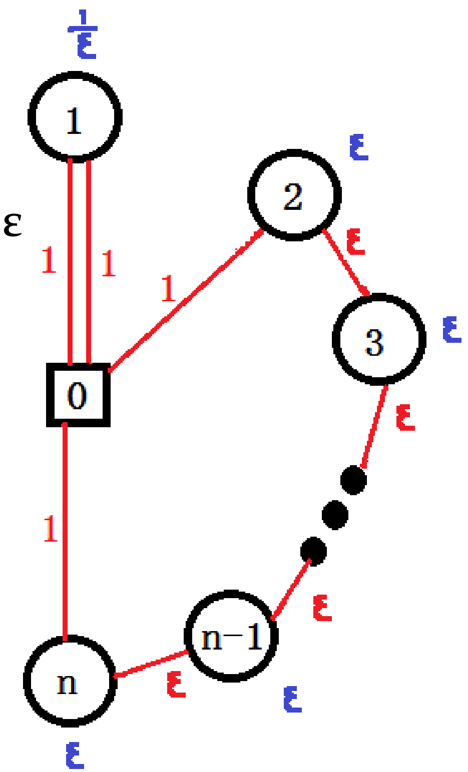
CVRP_INF

$$z_{MM}^{\infty} / z_{MS}^{\infty} \to \infty$$



customers: n =1+1/ε

capacity = 1/ε



$$z_{MM}^{\infty} = 2(1+1/\varepsilon)$$

$$z_{MS}^{\infty} = 4+(n-2)\varepsilon$$
$$= 5-\varepsilon$$

CVRP_k

- Capacitated VRP with at most *k* vehicles available

- $$r_{MS}^{(k)} \le z_{MS}^{(k)} \le z_{MM}^{(k)} \le k r_{MM}^{(k)}$$

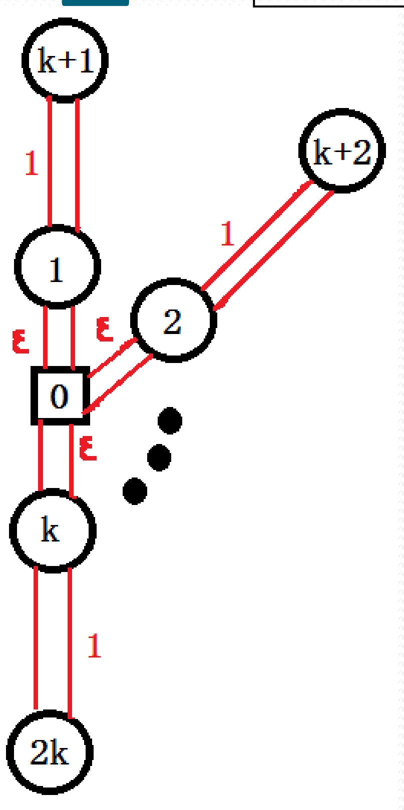
$$\Rightarrow r_{MS}^{(k)} / r_{MM}^{(k)} \le k$$

- $$z_{MM}^{(k)} \le k r_{MM}^{(k)} \le k r_{MS}^{(k)} \le k z_{MS}^{(k)}$$

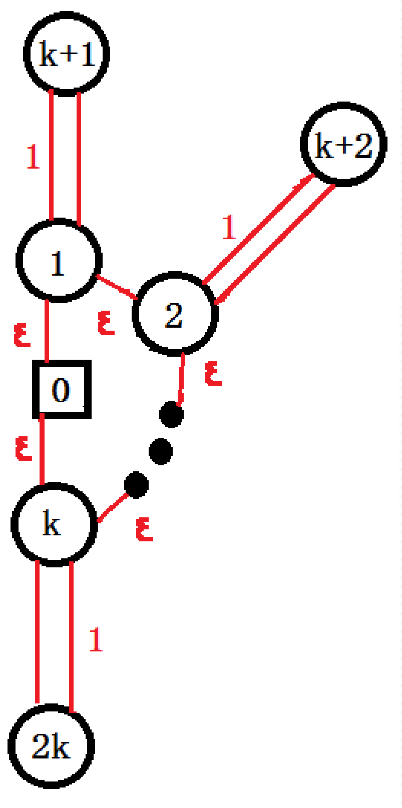
$$\Rightarrow z_{MM}^{(k)} / z_{MS}^{(k)} \le k$$

CVRP_k

$$\boxed{r_{MS}^{(k)} / r_{MM}^{(k)} \leq k}$$

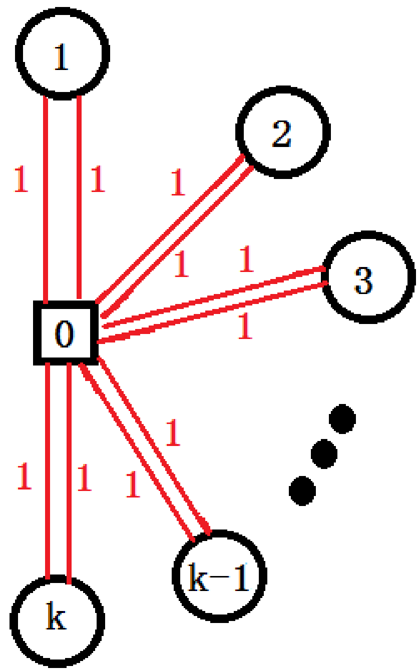


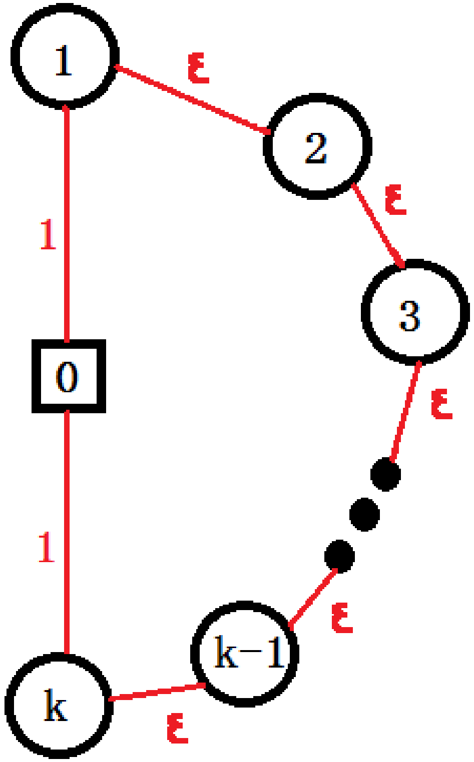
$$r_{MM}^{(k)} = 2 + 2\varepsilon$$



$$r_{MS}^{(k)} = 2k + (k+1)\varepsilon$$

CVRP_k $\boxed{z_{MM}^{(k)} / z_{MS}^{(k)} \leq k}$





$$z_{MM}^{(k)} = 2k$$

$$z_{MS}^{(k)} = 2 + (k-1)\varepsilon$$

MTSP_k

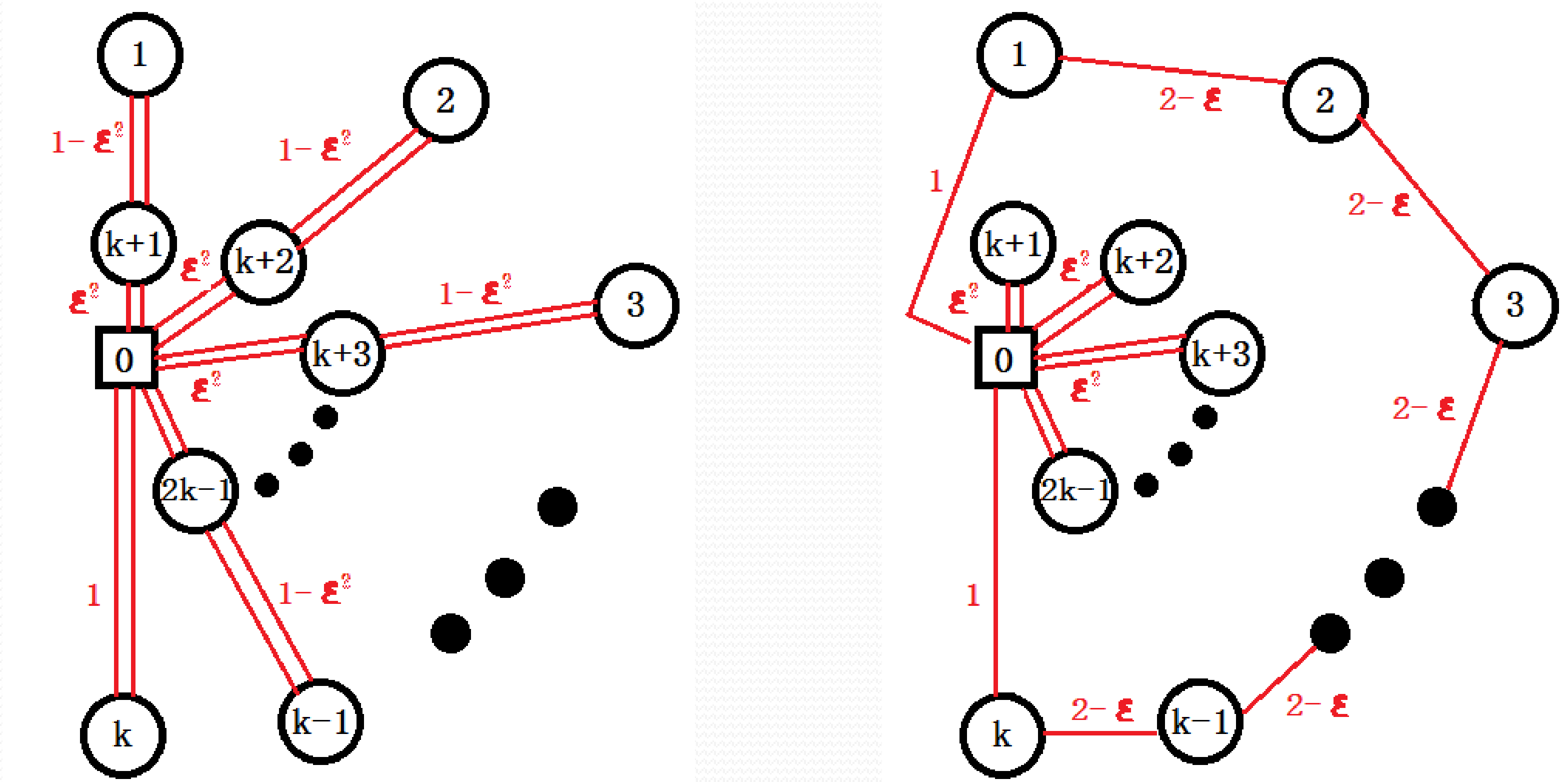
- Multiple TSP with *k* vehicles
  - The customers just have to be visited
  - Exactly *k* routes have to be defined

- $r_{MS}^{(M)} \le z_{MS}^{(M)} \le z_{MM}^{(M)} \le k r_{MM}^{(M)} \Rightarrow r_{MS}^{(M)} / r_{MM}^{(M)} \le k$

- $z_{MM}^{(M)} \le k r_{MM}^{(M)} \le k r_{MS}^{(M)} \le k z_{MS}^{(M)} \Rightarrow z_{MM}^{(M)} / z_{MS}^{(M)} \le k$

MTSP_k

$$\boxed{r_{MS}^{(M)} / r_{MM}^{(M)} \le k}$$

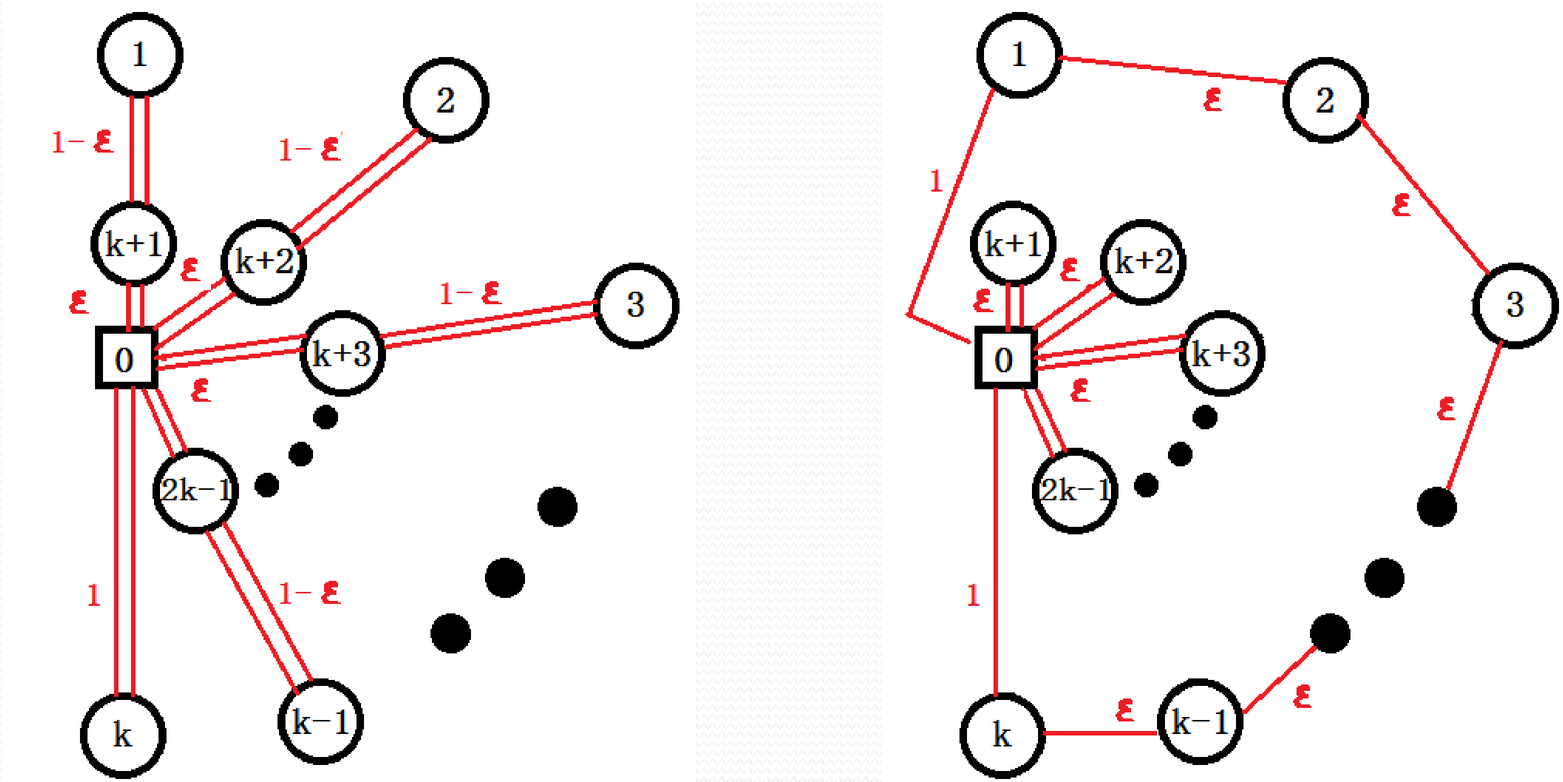


$$r_{MM}^{(M)} = 2$$

$$r_{MS}^{(M)} = 2 + (k-1)(2-\varepsilon) = 2k - (k-1)\varepsilon$$

MTSP_k

$$\boxed{z_{MM}^{(M)} / z_{MS}^{(M)} \le k}$$



$$z_{MM}^{(M)} = 2k$$

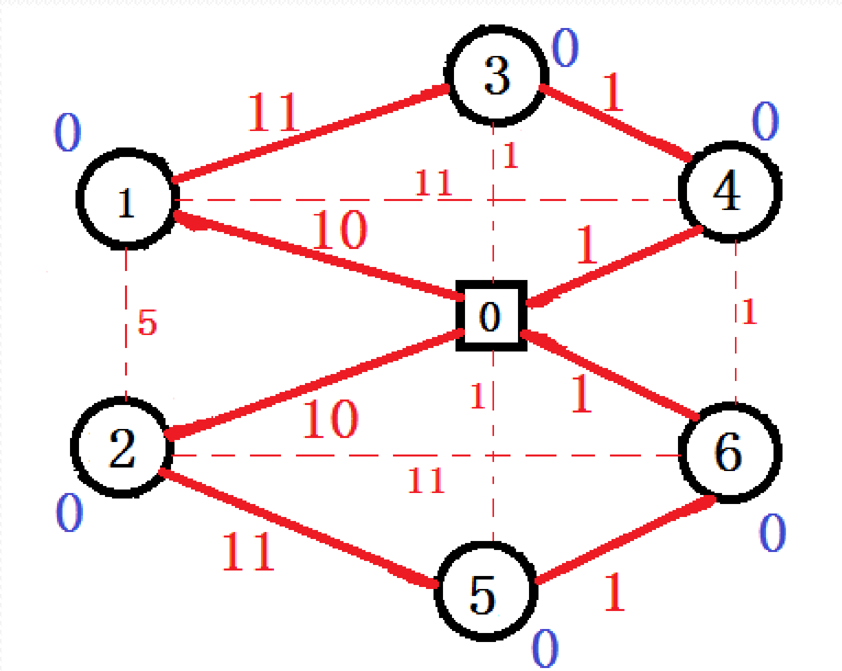
$$z_{MS}^{(M)} = 2 + 2(k-1)\varepsilon + (k-1)\varepsilon$$
$$= 2 + 3\varepsilon(k-1)$$

SVRP_k

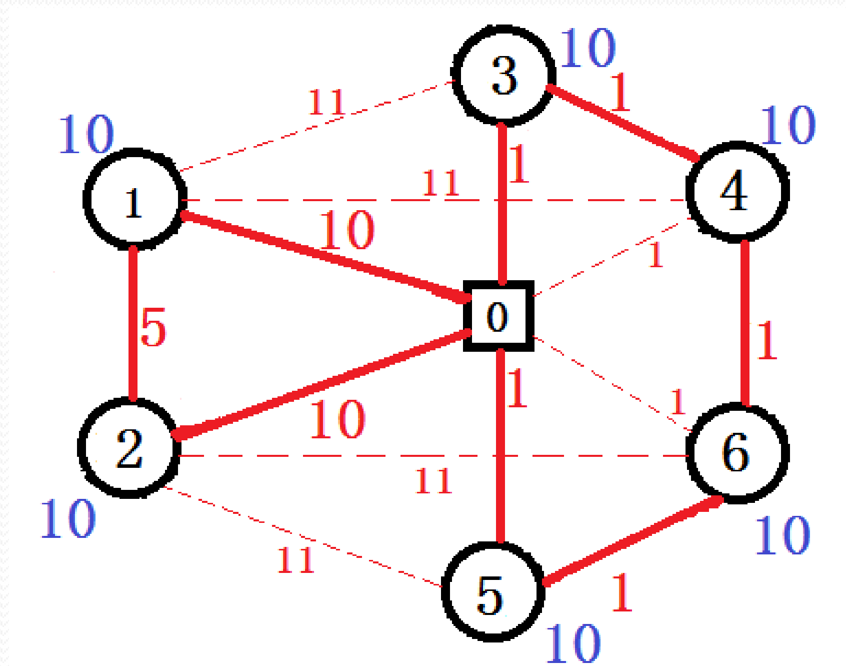
- Service time VRP with at most *k* vehicles
  - Customer demands are given in terms of service times
  - Cost of a route = travel time + service time
- Routing of the min-sum solution is not affected by service times
- Routing of the min-max solution may be affected by service times
- $r_{MS}^{(S)} \le z_{MS}^{(S)} \le z_{MM}^{(S)} \le k r_{MM}^{(S)} \Rightarrow r_{MS}^{(S)} / r_{MM}^{(S)} \le k$

SVRP_k

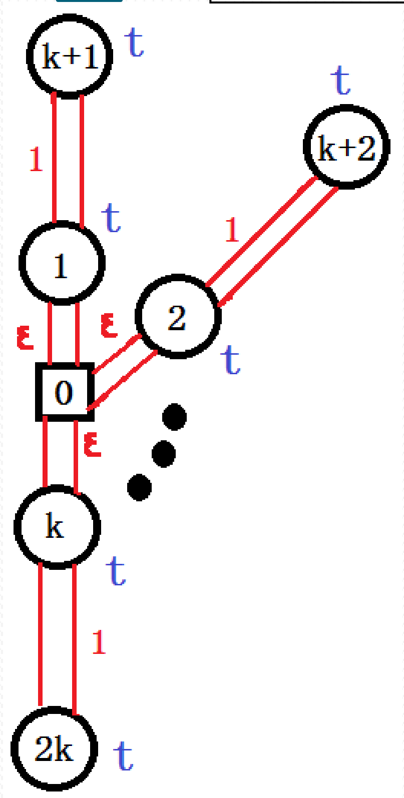
Min-max solution without service times

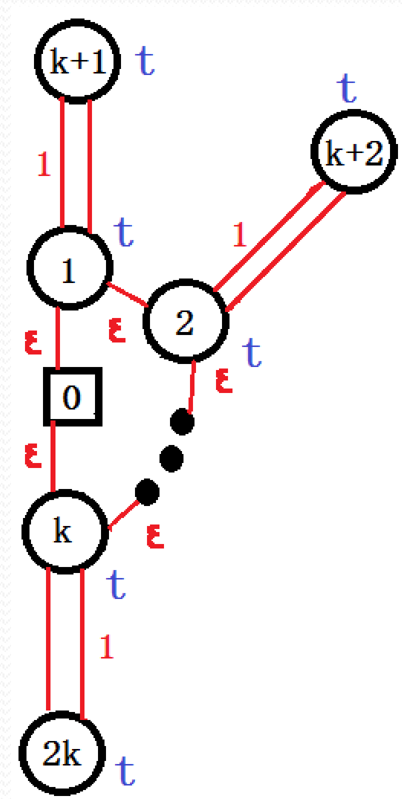


Min-max solution with service times



SVRP_k $\boxed{r_{MS}^{(S)} / r_{MM}^{(S)} \le k}$





$$r_{MM}^{(S)} = 2 + 2t + 2\varepsilon$$

$$r_{MS}^{(S)} = 2k + 2kt + (k+1)\varepsilon$$

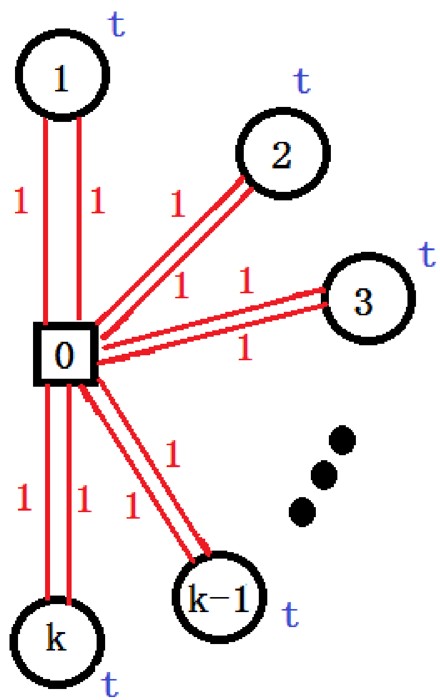
SVRP _k

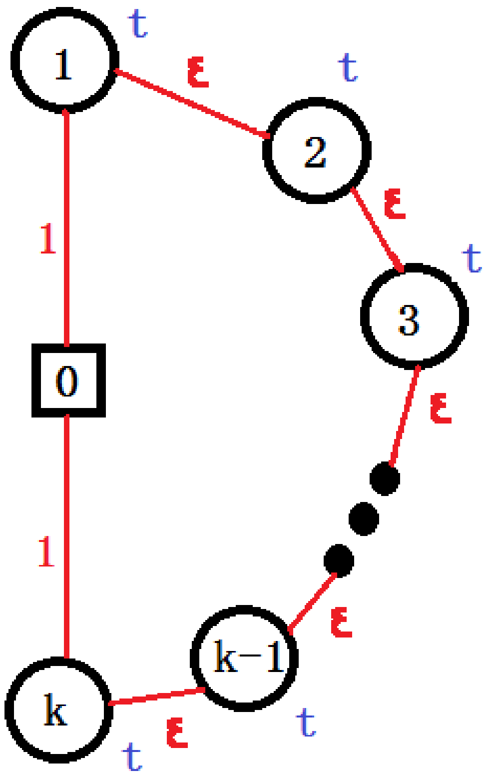
- The bound $z_{MM}^{(S)} \le kr_{MM}^{(S)} \le kr_{MS}^{(S)} \le kz_{MS}^{(S)} \Rightarrow z_{MM}^{(S)} / z_{MS}^{(S)} \le k$ is still valid, but no longer tight

- We prove the tight bound

  $z_{MM}^{(S)} \le kz_{MS}^{(S)} - (k-1)T$, where T = total service time in our paper

SVRP_k $\boxed{z_{MM}^{(S)} \le k z_{MS}^{(S)} - (k-1)T}$





$$z_{MM}^{(S)} = 2k + T$$

$$z_{MS}^{(S)} = 2 + \varepsilon(k-1) + T$$

A Summary

	Ratio of the cost of the longest route	Ratio of the total cost
CVRP_INF	$r_{MS}^{\infty} / r_{MM}^{\infty} \rightarrow \infty$	$z_{MM}^{\infty} / z_{MS}^{\infty} \rightarrow \infty$
CVRP_k	$z_{MM}^{(k)} / z_{MS}^{(k)} \leq k$	$z_{MM}^{(k)} / z_{MS}^{(k)} \leq k$
MTSP_k	$r_{MS}^{(M)} / r_{MM}^{(M)} \leq k$	$z_{MM}^{(M)} / z_{MS}^{(M)} \leq k$
SVRP_k	$r_{MS}^{(S)} / r_{MM}^{(S)} \leq k$	$z_{MM}^{(S)} \leq k z_{MS}^{(S)} - (k-1)T$

Conclusions

- If your true objective is min-max, don’t use the min-sum solution
- If your true objective is min-sum, don’t use the min-max solution